MATHEMATICS

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A Textbook for Class IX

Part I

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Foreword

The National Policy on Education (NPE) 1986 has emphasized the need for qualitative improvement of school education, particularly in the area of science and mathematics education. The Government of India has already initiated a number of steps in this direction. The National Council of Educational Research and Training (NCERT) has been assigned the responsibility of developing a new curriculum and related curricular materials in line with the new education policy to serve as models for the States and the Union Territories to adopt/adapt.

The NCERT has been working in the area of curriculum development/renewal for many years. Yet developing a curriculum in tune with the intentions and aspirations of NPE poses a real challenge. The various curricular issues arising out of the NPE were discussed in a number of seminars/workshops resulting in the Council's document *National Curriculum for Elementary and Secondary Education – A Framework* (Revised version 1987) and several other documents.

Science and mathematics are the vital areas of school curriculum. So, the NCERT thought it appropriate to make the best use of the available expertise in the country in science and mathematics education in the process of curriculum renewal and preparation of new generation of instructional materials Accordingly, a General Advisory Board for Science and Mathematics was constituted under the chairmanship of Prof. C. N. R. Rao, an eminent scientist and Chairman of the Prime Minister's Scientific Advisory Committee. On the advice of this General Advisory Board, six writing teams were set up for developing new instructional packages in science and mathematics in line with NPE The writing team for mathematics was constituted under the chairmanship of Prof. U. N. Singh The team consists of distinguished mathematicians from various universities, besides NCERT experts. The writing team met several times and after much deliberations evolved a new mathematics curriculum which, I am sure reflects the intentions and aspirations of NPE. The present textbook is based on the new curriculum.

The authors spared no efforts to produce materials of high quality First, the draft materials prepared by different authors were continuously refined through mutual discussions within the group. Then the materials were exposed to a group of classroom teachers drawn from all over the country in a review workshop. The suggestions and comments made in the review workshop were incorporated by the authors as far as possible and finally the whole manuscript was edited by Prof. U. N. Singh.

I am indeed very thankful to Prof. U. N. Singh, Prof. A. M. Vaidya, Prof. R. Vittal Rao, Prof. V. G. Tikekar, Prof. J. D. Gupta, Prof. K. V. Rao, Dr. Ram Autar and Shri Ishwar Chandra who authored different parts of the book. I am particularly grateful to Prof. U. N. Singh, who provided valuable guidance to his team of authors and finally edited the whole manuscript. I express my deep appreciation to my colleagues in the Department of Education in Science and

Mathematics, Prof. K. V Rao, Dr Ram Autar, Shri Ishwar Chandra, who took a lot of pains in shaping the manuscript in the press-worthy form and seeing it through the press. I am very much indebted to the teachers who participated in the review workshop and provided valuable suggestions and comments for the improvement of the draft materials. I must make a special mention of Prof. A. K. Jalaluddin, Joint Director, NCERT, and Prof B Ganguly, Head, Department of Education in Science and Mathematics, who took a lot of interest in this project and greatly helped in bringing out this book. I also express my thanks to Shri C. N Rao, Head, Publication Department, and his Publication Team for making all efforts in bringing out this book expeditiously and in an excellent form.

Curriculum development is a dynamic and continuous process. No one can claim to have developed a perfect curriculum or perfect curricular materials Though Prof. U. N. Singh and his able team of mathematicians did a very good job of evolving a new mathematics curriculum and new curricular materials in line with NPE, there will always be some scope for further improvement. So, I request all those who will be using this book to evaluate the materials with an open mind and offer their valuable suggestions for further improvement

P L. MALHOTRA
Director
National Council of
Educational Research and Training

Preface

The National Policy on Education (NPE-1986) has justifiably emphasised the need for qualitative improvement in school education in Science and Mathematics. The following two paragraphs of the NPE deserve special attention in this connection.

> 8 16 Mathematics should be visualized as the vehicle to train a child to think, reason, analyse and articulate logically. Apart from being a specific subject it should be treated as concomitant to any subject involving analysis and reasoning.
>
> 8 17 With the recent introduction of computers in schools educational computing and emergence of learning through the cause-effect relationships and the interplay of variables, the teaching of mathematics will be suitably redesigned to bring it in line with modern technological devices.

The implementation of the NPE made it necessary to review the courses of studies in science and mathematics and to bring out new textbooks. There is another strong reason for reviewing the course contents in science and mathematics and for rewriting textbooks in these disciplines. Different branches of science, pure and applied, including mathematics are developing with astonishing rapidity. Exciting discoveries of far-reaching importance are being made in quick succession. Deep and new ideas of a rapidly growing science very often shed new and penetrating light even on the most elementary topics. It is, therefore, highly desirable that courses of studies of school education are reviewed periodically and new textbooks are written

The National Council of Educational Research and Training initiated prompt action in respect of redesigning the curricula in science subjects and mathematics relating to school education. It appointed a General Advisory Board for Science and Mathematics under the Chairmanship of Professor C. N. R. Rao, who is also the Chairman of the Scientific Advisory Committee of the Prime Minister. On the advice of General Advisory Board, the NCERT constituted six writing teams for developing instructional packages in science subjects and mathematics from upper primary to senior secondary level. Besides the experts from the NCERT, distinguished mathematicians from different parts of the country are members of the writing team in mathematics.

The NCERT had done a good deal of preparatory work in connection with curriculum renewal before the appointment of the writing team. We have been greatly benefited in our work by the NCERT's documents:

- National Curriculum for Elementary and Secondary Education – A Framework
- Mathematics education for the first 10 years of schooling – Guidelines for developing curriculum for upper primary and high school stages
- Draft syllabi in mathematics for upper primary, secondary, and senior secondary levels

The present textbook has been written on the basis of a curriculum which emerged after a thorough review of the curriculum prepared by the NCERT.

A new textbook should be written only when it has to say new things or give a new message I believe that the present book has some new ideas. Some special features of this book are as follows

1. In proving results deductive reasoning has been emphasized especially in the chapters on geometry If the students learn and appreciate the power of deductive reasoning, it will certainly help them to develop critical power of analysis and reasoning
2. The book contains material for different categories of students: (a) those for whom secondary education is a terminal point, (b) those who are going to continue the study of mathematics for using it as a tool in the study of other disciplines or in their professions (c) those who have special aptitude and talents for studying mathematics and are going to be research mathematicians
3. The distinction between rational and irrational numbers has been clearly explained by using decimals, and the representation of real numbers by points on a line has been simply explained.
4. A chapter on elementary statistics and another on computing have been written in simple style. These chapters are meant to stimulate the interest of students in pursuing the study of statistics and computer programming.
5. New concepts have been introduced by means of simple examples

Our group is also working on the development of additional instructional materials to supplement the textbook. The additional materials are Supplementary Problem Book, Enrichment Mathematics, Teachers' Guide, and so on. I hope these additional materials will soon be made available to the students and teachers

The first draft of the book was exposed to a group of school teachers teaching class IX in schools in different parts of the country in a review workshop organized by the NCERT at Delhi. The school teachers made important suggestions which were incorporated in the second draft I thank the school teachers for their suggestions.

I am thankful to Dr. P. L. Malhotra, Director, NCERT who initiated this project and invited us to join this national endeavour for the improvement of mathematics education.

I am grateful to Prof C. N. R. Rao for his constant guidance which helped us in planning and developing this textbook. I express my sincere thanks to Prof. A. K. Jalaluddin, Joint Director, NCERT, and Prof. B. Ganguly, Head, Department of Education in Science and Mathematics, for their kind cooperation extended to me and to the writing team.

Prof. A. M Vaidya, Prof. R Vittal Rao, Prof. V. G. Tikekar, Prof. J. D Gupta, Prof. K. V. Rao, Dr Ram Autar and Shri Ishwar Chandra were my colleagues in the writing team All of them most willingly spared their valuable time for preparing the book I express my deep sense of gratitude to them. Prof. J. D. Gupta was invited by me later. In addition to the writing part of the book, Prof. J. D Gupta and Prof. K. V. Rao also helped me in the editing part of the

book. I thank them for their cooperaion. Dr. B L. Sharma of Allahabad University and Dr. Satyadeo of Jammu University joined us in the beginning, but both of them withdrew subsequently as they had to go abroad. They were with us during the initial stages of preparing the first draft and gave valuable advice. I express my thanks to both of them Besides taking part in the writing work, Prof. K. V Rao, Dr Ram Autar, Shri Ishwar Chandra of NCERT had to put in hard work in organizing several workshops, getting the manuscript in pressworthy form and finally seeing it through the Press. I am indeed very thankful to them.

In spite of the great care taken by the authors and the NCERT team, some errors may have escaped our notice We shall appreciate it very much if such errors are brought to our notice. Suggestions for improving the quality of the book will be gratefully received

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Chairman of the Writing Team

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Contents

CHAPTER 1

The Language of Sets

1.1 Sets

The concept of **Set** is of fundamental importance in mathematics This concept is fairly simple, and we come across sets of different kinds in everyday life. A football team is a set of players, a class is a set of students. The school library houses a set of books in mathematics, a set of books in physics, and so on There is a set of exercises in each chapter in this book. Thus, we can say that by a set we mean a collection of objects. In this book, we will be mainly dealing with sets of mathematical objects like numbers, points, lines, triangles, circles, and so on. We usually denote sets by capital letters, A, B, C, S, X, Y, etc; and the objects of a set by lower-case letters, a, b, c, etc.

If S is any set, every object in S is also called an element of the set. Let S be the set of all natural numbers less than 100 Then 10 is an element of this set 42 is also an element of this set. Can you name some more elements of the set? Is 100 an element of the set? The fact that 10 is an element of the set S is expressed in symbols as "$10 \in S$" which is read as "10 belongs to S" or "10 is an element of S." Similarly, "$42 \in S$." As we see, 108 is not an element of the above set S. We express this fact in symbols as "$108 \notin S$." In general, if a is an element of a set S, we just write "$a \in S$," and read it as "a is an element of S" If a is not an element of S, we write "$a \notin S$," which is read as "a is not an element of S." ("$a \in S$" is also read as "a belongs to S," and "$a \notin S$" is also read as "a does not belong to S")

Usually, one of the following two ways of describing a set are used

(1) *Roster Form* (also known as *Tabular form*);
(2) *Set Builder Form* (also known as *Rule form*).

Roster Form

In roster form all the elements of the set are listed, the elements being separated by commas, and are enclosed within braces. For example, the set of all even positive integers less than 7 is described in roster form as "{2, 4, 6}." While describing a set in roster form you must bear in mind that every element is listed only once and that the order in which the elements are listed is immaterial Thus, {2, 6, 4} or {6, 2, 4} also describe the set {2, 4, 6}. But {2, 4, 6, 2} is not a proper description of the above set, since the element 2 has been listed twice Also, {2, 4} is not a proper description of the above set, since the element 6 has been omitted

Example 1.1: The set of all natural numbers that divide 42 can be described, in roster form, as

$\{1, 2, 3, 6, 7, 14, 21, 42\}$

Can it also be described as

$\{1, 3, 7, 21, 2, 6, 14, 42\}$? (Why?)

Can it be described as

$\{1, 3, 7, 21\}$? (Why?)

Can it be described as

$\{1, 3, 7, 7, 21, 2, 6, 14, 42\}$? (Why?)

Example 1.2: Given below are some sets in roster form

(*i*) $\{2, 4, 6, 8\}$ is the set of positive even integers less than 10

(*ii*) $\{a, e, i, o, u\}$ is the set of vowels in the English alphabet.

(*iii*) $\{I, N, D, A\}$ is the set of letters forming the word "INDIA."

(*iv*) $\{S, C, H, O, L\}$ is the set of letters forming the word "SCHOOL."

The roster form enables us to see all the elements of a set at a glance However, it is not convenient to use when the number of elements of a set is large. In such cases it is more convenient to use the set builder form which is explained below.

Set Builder Form

This form is used when all the elements of a set possess a single common property which is not possessed by any element outside the set For example, in the set "$\{a, e, i, o, u\}$" all the elements possess a common property, namely, each of them is a vowel in the English alphabet, and no other letter possesses this property. Denoting this set by V, we write

$V = \{x : x$ is a vowel in the English alphabet.$\}$

It may be observed that we describe the set by using a symbol x (any other symbol like the letters y, z, etc., could be used) which is followed by a colon ":" After the sign of colon we write the characteristic property possessed by the elements of the set and then enclose the whole description within braces. The above description of the set V is read as "The set V of all x such that x is a vowel of the English alphabet." In this description the braces stand for "the set of all"; the colon stands for "such that."

For example, the following description of a set

$A = \{x : x$ is a natural number and $3 < x < 10\}$

is read as "the set of all x such that x is a natural number and $3 < x < 10$." Hence the numbers 4, 5, 6, 7, 8 and 9 are the elements of the set A.

Example 1.3: Suppose a set E is given as

$E = \{x : x$ is a positive even integer $\}$

This means that E is the "set of all x such that x is a positive even integer."

Example 1.4: Let l be a line in the plane. If we write

$P = \{m : m$ is a line parallel to l $\}$

then P is the set of all lines in the plane that are parallel to l.

Let us consider the set of all integers satisfying the inequality $5 \leq x \leq 6$. We can write this set in roster form as $\{5, 6\}$. Now consider the set of all integers satisfying the inequality $5 < x < 6$ We find that no integer satisfies this inequality, and hence this set has no elements. It is convenient to introduce the notion of a set having no elements We call such a set, the "empty set", "null set" or "void set" and denote it by the symbol '$\varnothing$' (Sometimes the empty set is written in roster form as "{ }").

Two sets are said to be equal if they have the same elements

The sets $A = \{a, b, c\}$, $B = \{b, a, c\}$ are equal sets

Similarly sets $X = \{1, 2, 3, 4\}$ and $Y = \left\{\frac{2}{2}, \frac{4}{2}, \frac{6}{2}, \frac{8}{2}\right\}$ are equal sets In other words, two equal sets are essentially identical.

Finite and Infinite Sets

Let

$A = \{1, 2, 3, 4, 5\}$
$B = \{a, b, c, d, e, f\}$
$C = \{$men living in different parts of the world today$\}$.

As can be seen readily, A contains 5 elements and B contains 6 elements. How many elements does C contain? As it is, we do now know the number of elements in C, but it is some definite number; may be quite a big number Sets which contain a definite number of elements are called "finite sets."

Consider the set of natural numbers How many elements does this set contain? Is there a natural number beyond which there is no other natural number? (No.) You see, therefore, that the number of elements of this set is not finite like the number of elements in A, B or C we considered above. We say the set of natural numbers is an "infinite set." We will come across many infinite sets in our study hereafter. For instance, the set of even numbers, the set of odd numbers, the set of rational numbers are all infinite sets.

When we represent a set in the roster form, we write all the elements of the set within { }. It is not possible to write all the elements of an infinite set within { } because the number of elements of such a set is not finite So we represent an infinite set in the roster form by writing a few elements which clearly indicate the structure of the set, followed (or preceded) by three dots.

For instance, $\{1, 2, 3, 4, \ldots\}$ is the set of natural numbers.
$\{1, 3, 5, 7, 9, \ldots\}$ is the set of odd natural numbers.
$\{\ldots, -3, -2, -1, 0, 2, 3, \ldots\}$ is the set of integers.

Exercises 1.1

1. Let $A = \{1, 2, 3, 4, 5, 6\}$. Insert the appropriate symbol $\in$ or $\notin$ in the blank spaces:
   (i) $5 \ldots A$ (ii) $8 \ldots A$

(iii) 0 ... A (iv) 4 ... A

(v) 2 ... A (vi) 10 .. A

2. Write the following sets in roster form:
   (i) A = $\{x : x$ is an integer and $-3 \leq x < 7.\}$
   (ii) B = $\{x . x$ is a natural number less than 6 $\}$
   (iii) C = $\{x : x$ is a two-digit natural number such that the sum of its digits is 8.$\}$
   (iv) D = $\{x : x$ is a prime number which is a divisor of 60.$\}$
   (v) E = The set of all letters in the word MATHEMATICS.
   (vi) F = The set of all letters in the word SETS.
3. Write the following sets in the set builder form:
   (i) $\{3, 6, 9, 12\}$ (ii) $\{2, 4, 8, 16, 32\}$
   (iii) $\{5, 25, 125, 625\}$ (iv) $\{a, e, i, o, u\}$
   (v) $\{1, 3, 5, ..\}$ (vi) $\{2, 4, 6...\}$
   (vii) $\{1, 4, 9, .. ,100\}$
4. Match each of the sets on the left described in the roster form with the same set on the right described in set builder form:
   (i) $\{1, 2, 3, 6\}$ (a) $\{x : x$ is a prime number and a divisor of 6.$\}$
   (ii) $\{2, 3\}$ (b) $\{x : x$ is an odd natural number less than 10.$\}$
   (iii) $\{H, A, Y, R, N\}$ (c) $\{x . x$ is a natural number and divisor of 6.$\}$
   (iv) $\{1, 3, 5, 7, 9\}$ (d) $\{x : x$ is a letter of the word 'HARYANA'.$\}$
5. In the following, state whether $A = B$ or not:
   (i) $A = \{1, 2, 3, 4\}$ $B = \{4, 3, 2, 1\}$
   (ii) $A = \{4, 8, 12, 16\}$ $B = \{8, 4, 16, 18\}$
   (iii) $A = \{2, 4, 6, 8, 10\}$ $B = \{x : x$ is positive even integer $\leq 10.\}$
   (iv) $A = \{x : x$ is a multiple of $10\}$ $B = \{10, 15, 20, 25, 30, ...\}$

1.2 Subset

We may call a part of a set A as a subset of A. For example, let Z denote the set of all integers (positive, negative and zero) Let N be the set of all natural numbers. Then N is a part of Z. We say that N is a subset of Z The fact that N is a subset of Z is expressed in symbols as $N \subset Z$. The symbol $\subset$ stands for "is a subset of" or "is contained in." Thus we arrive at the following definition:

A set A is said to be a subset of a set B if every element of A is also an element of B.

In other words, $A \subset B$ if whenever $a \in A$ then $a \in B$.

It is often convenient to use the symbol "$\Rightarrow$" which means "implies." Using this symbol we can write the definition of "subset" as follows:

$A \subset B$ if $a \in A \Rightarrow a \in B$.

We read the above as "A is a subset of B if the fact that a is an element of A implies that a is also an element of B."

It follows from this definition that $A \subset A$ for every set A. Since the empty set $\varnothing$ has no elements, we agree to say that $\varnothing \subset A$.

Example 1.5: Let A be the set of all the students of the Class 9 in your school and B be the set of all the students studying in your school. Then A is a subset of B, and we write $A \subset B$.

Example 1.6: If A is the set of all divisors of 56 and B the set of all prime divisors of 56, then B is a subset of A, and we write $B \subset A$.

Example 1.7: Let $A = \{1, 3, 5\}$ and $B = \{x : x$ is an odd natural number less than $6\}$. Is A a subset of B? Yes. (Why?). Is B a subset of A? Yes. (Why?). Is $A = B$?

In Example 1.7 you find that two given sets A and B are such that A is a subset of B and B is a subset of A, and that $A = B$. Can you generalize this? It is always true that, if A is a subset of B and B is a subset of A, then $A = B$ Conversely, if A is equal to B, then A is a subset of B and B is a subset of A. In symbols,

$A = B$, if and only if $A \subset B$ and $B \subset A$.

Example 1.8: Let A be the set of all factors of 42, B the set of all factors of 60 and C the set $\{1, 2, 3\}$. Then you can easily verify that $C \subset A$ and also $C \subset B$
You may also verify that C is the set of common factors of 42 and 60.

Example 1.9: Let $A = \{a, e, i, o, u\}$, $B = \{a, b, c, d\}$. Is A a subset of B? No. (Why?). Is B a subset of A? No (Why?).

When A is not a subset of B we write this in symbols as $A \not\subset B$

Example 1.10: Let us write down all the subsets of the set $\{1, 2\}$. We know $\varnothing$ is a subset of every set. So $\varnothing$ is a subset of $\{1, 2\}$. We see that $\{1\}$ and $\{2\}$ are also subsets of $\{1, 2\}$ Also we know every set is a subset of itself. So $\{1, 2\}$ is a subset of $\{1, 2\}$. Thus the set $\{1, 2\}$ has, in all, four subsets, *viz* $\varnothing$, $\{1\}$, $\{2\}$ and $\{1, 2\}$

Exercises 1.2

1. If $A = \{a, b, c\}$, and $B = \{b, c, a, d\}$, is $A \subset B$? Is $B \subset A$?
2. Make correct statements by filling in the symbols $\subset$ or $\not\subset$ in the blank spaces:
   (*i*) $\{2, 3, 4\}$ ... $\{1, 2, 3, 4, 5\}$
   (*ii*) $\{a, b, c\}$ ... $\{b, c, d\}$
   (*iii*) $\{x : x$ is a student of Class 9 of your school.$\}$ ... $\{x : x$ is a student of your school$\}$
   (*iv*) $\{x : x$ is a circle in the plane.$\}$ ... $\{x : x$ is a circle with radius 1.$\}$
   (*v*) $\{x : x$ is a triangle in the plane.$\}$ ... $\{x : x$ is a rectangle in the plane$\}$
   (*vi*) $\{x : x$ is an equilateral triangle in the plane$\}$. $\{x : x$ is a triangle in the plane.$\}$
   (*vii*) $\{x : x$ is an even natural number.$\}$ ... $\{x : x$ is an integer.$\}$

3. Examine whether the following statements are true or false:
   (i) $\{a, b\} \not\subset \{b, c, a\}$
   (ii) $\{a, e\} \not\subset \{x . x$ is a vowel in the English alphabet.$\}$
   (iii) $\{1, 2, 3\} \not\subset \{1, 2, 3\}$
   (iv) $\{a\} \not\subset \{a, b, c\}$
   (v) $\{a\} \in \{a, b, c\}$
   (vi) $\{x . x$ is an even natural number less than $6\}$ $\subset \{x : x$ is a natural number which divides $36\}$
4. Let

$$X = \{0, 1, -1, 2, -2\}; Y = \{0, 1, 2\};$$
$$A = \{x : x \text{ is a natural number.}\}$$
$$B = \{x : x \text{ is an integer.}\}$$
$$C = \{x : x \text{ is a natural number less than } 3\}$$
$$D = \{1, 2, 3\}$$
$$E = \{1, 2\}$$

Which of the following statements are true and which are false?

(i) $X \subset A$ (ii) $A \subset X$ (iii) $X \subset B$
(iv) $B \subset X$ (v) $X \subset C$ (vi) $C \subset X$
(vii) $X \subset D$ (viii) $D \subset X$ (ix) $X \not\subset E$
(x) $E \not\subset X$ (xi) $D = E$ (xii) $C \not\subset D$
(xiii) $B \subset D$ (xiv) $D \subset E$ (xv) $E \subset D$
(xvi) $\varnothing \subset X$ (xvii) $X \subset \varnothing$ (xviii) $\varnothing \subset A$

5. Write down all the subsets of the following sets:
   (i) $\{a\}$ (ii) $\{a, b\}$
   (iii) $\{1, 2, 3\}$ (iv) $\varnothing$

1.3 Operations on Sets

We are familiar with the arithmetical operations of addition, subtraction, multiplication and division. Each one of these operations was performed on a pair of numbers to get another number. For example, when we perform the operation of addition on the pair of numbers 2 and 3 we get the number 5. Again, performing the operation of multiplication on the pair of numbers 2 and 3 we get 6, since $2 \times 3 = 6$. Similarly, there are some operations which when performed on two sets give rise to another set. We will now define some of these operations on sets and study their properties.

Union of Sets

Let A and B be any two sets. By the union of A with B we mean the set which consists of all the elements of A and all the elements of B, the common elements being taken only once. We write the union of A with B as $A \cup B$ and read it as "A union B."

Example 1.11: Let $A = \{1, 2, 3\}$ and $B = \{2, 3, 5\}$. Then $A \cup B = \{1, 2, 3, 5\}$. Note that the common elements 2 and 3 have been taken only once while writing $A \cup B$

Example 1.12: Let $H = \{\text{Ram, Shyam, Akbar}\}$ be the set of students of Class 9, who are in the school hockey team. Let $F = \{\text{Shyam, David, Ashok}\}$ be the set of students from Class 9, who are in the school football team. Then,

$$H \cup F = \{\text{Ram, Shyam, Akbar, David, Ashok}\}.$$

This is the set of students from Class 9 who are in the hockey team or the football team or both.

Thus,

The union of two sets A *and* B *is the set* C *which consists of all those elements which are either in* A *or in* B *or in both.* In symbols, we write

$$A \cup B = \{x : x \in A \text{ or } x \in B \text{ or } x \in \text{ both } A \text{ and } B\}.$$

In other words, we say that $A \cup B = \{x : x$ belongs to at least one of the sets A and $B\}$

Example 1.13. Let $A = \{1, 2, 3\}$ and $B = \{3, a, b\}$ Let us find $A \cup B$ and $B \cup A$?

$$A \cup B = \{1, 2, 3, a, b\}$$
$$B \cup A = \{3, a, b, 1, 2\}$$

You can easily see that $A \cup B = B \cup A$, since they consist of the same elements.

In general, for any two sets A and B,

$$A \cup B = B \cup A$$

and this follows from the definition of the union of sets. This is known as the *Commutative Law* for the union of two sets

Example 1.14: Let $A = \{1, 2, 3\}$. Let us find $A \cup A$

$$A \cup A = \{1, 2, 3\} \quad \text{(Why?)}$$

Thus,

$$A \cup A = A$$

Again, it follows from the definition of "union" that $A \cup A = A$ for any set A.

Intersection of Sets

When we draw two circles in the plane what do we mean by their intersection? We mean the set of points which are in both the circles. Similarly, if A and B are any two sets we mean by the intersection of A with B, the set of all elements common to both A and B. We write the intersection of A with B in symbols as $A \cap B$ which is read as "A intersection B." We have

$$A \cap B = \{x : x \in A \textbf{ and } x \in B\}$$

Example 1.15: Consider the sets A and B of Example 1.11, You get

$$A \cap B = \{2, 3\}$$

Example 1.16: Consider the sets H and F of Example 1.12. You get

$$H \cap F = \{\text{Shyam}\}$$

Example 1.17: Consider the sets A and B of Example 1.13 You get

$$A \cap B = \{3\}$$

Example 1.18: For the set A of Example 1.14, we have

$$A \cap A = A$$

The result of Example 1.18 is true for any set A. That is to say that

$A \cap A = A$ for any set A

Example 1.19: Let $A = \{1, 2, 3\}, B = \{4, 5, 6\}$. Let us find $A \cap B$

$A \cap B$ is the set of all those elements which are common to both A and B. Since A and B have no common elements, their intersection $A \cap B$ has no elements and as a result $A \cap B$ is the empty set. Hence,

$$A \cap B = \varnothing$$

In Example 1.19 we have seen two sets whose intersection is the empty set

If *A and B are two sets such that $A \cap B = \varnothing$, then we say that the two sets are disjoint.*

Thus, in Example 1.19 the sets A and B are disjoint.

Example 1.20: Let us consider the sets H and F of Example 1.13. We have seen in Example 1.16 that $H \cap F = \{\text{Shyam}\}$ Thus $H \cap F$ is not empty, and so H and F are not disjoint.

Exercises 1.3

1. For the following sets find their union:

   (*i*) $A = \{a, e, i, o, u\}$; $B = \{a, b\}$

   (*ii*) $X = \{1, 3, 5\}$; $Y = \{1, 2, 3\}$

   (*iii*) $A = \{x : x \text{ is a natural number and multiple of 3.}\}$

   $B = \{x : x \text{ is a natural number less than 6 }\}$

   (*iv*) $A = \{x : x \text{ is a natural number and } 1 < x \leq 6.\}$

   $B = \{x : x \text{ is a natural number and } 6 < x < 10.\}$

   (*v*) $A = \{1, 2, 3\}, B = \varnothing$

2. Let $A = \{a, b\}, B = \{a, b, c\}$ *Is* $A \subset B$? What is $A \cup B$?
3. If A and B are two sets such that $A \subset B$ then what is $A \cup B$?
4. If A is any set, what is $A \cup \varnothing$?
5. Find the intersection of each pair of sets in parts (i), (ii), (iii) of Exercise 1 above.
6. Which of the following pairs of sets are disjoint?

   (*i*) $\{1, 2, 3, 4\}$ and $\{x : x \text{ is a natural number and } 4 \leq x \leq 6 \}$

   (*ii*) $\{a, e, i, o, u\}$ and $\{c, d, e, f\}$

   (*iii*) $\{x : x \text{ is an even integer.}\}$ and $\{x : x \text{ is an odd integer }\}$

7. For any two sets A and B, is it true to say that $A \cap B = B \cap A$?
8. Find $A \cap \varnothing$ for any set A.

1.4 Complement of a Set

Usually, in a particular context, we have to deal with the elements and subsets of a given set which is relevant to that particular context For example, while studying the theory of numbers, we are interested in the set of natural numbers and its subsets such as the subset of all prime numbers, the subset of all even numbers, and so forth In a particular context, we are interested in studying the properties of the subsets of a basic set. This basic set is called the "universal set."The universal set is usually denoted by U, and all its subsets by the letters A, B, C, etc.

Let U be the universal set of all prime numbers. Let A be that subset of U which consists of all those prime numbers that are not divisors of 42; thus $A = \{x : x \in \text{U and } x \text{ is not a divisor of } 42\}$. We see that $2 \in$ U but $2 \notin A$, because 2 is a divisor of 42. Similarly, $3 \in$ U but $3 \notin A$, and $7 \in$ U, but $7 \notin A$. Now 2, 3 and 7 are the only elements of U which do not belong to A. The set of these three prime numbers, i e., the $\{2, 3, 7\}$ is called the "Complement of A" with respect to U, and is denoted by A'. So we have

$$A' = \{2, 3, 7\}$$

Thus we see that

$$A' = \{x : x \in \text{U and } x \notin A\}.$$ This leads to the following definition.

Definition

If U is the universal set and A is a subset of U, then the complement of A with respect to U denoted by A' is defined as.

$$A' = \{x : x \in \text{U and } x \notin A\}$$

Where the universal set is clearly understood, we simply write A' is the complement of A, omitting the phrase "with respect of."

Example 1.21: Let U be the universal set of all the students of class 9 of your school. Let A be the set of all girls in the 9th class Then A' is the set of all boys in the 9th class.

Example 1.22: Let $U = \{1, 2, 3, 4, 5, 6\}$ and
$A = \{2, 4, 6\}$
Then $A' = \{1, 3, 5\}$.

If A is a subset of the universal set U, then its complement A' is also a subset of U. What is the complement of A'? That is, we have to find $(A')'$. In Example 1.22 above, we have

$$A' = \{1, 3, 5\}$$

Hence

$$\begin{aligned}(A')' &= \{x : x \in U \text{ and } x \notin A'\}\\ &= \{2, 4, 6\}\\ &= A.\end{aligned}$$

It is clear from the definition of the complement that for any subset A of the universal set U, we have

$$(A')' = A$$

Example 1.23: Let U = Set of all letters of the English alphabet.

Thus $U = \{a, b, c, d, .., x, y, z\}$.

Let $A = \{a, e, i, o, u\}$, i.e, A is the set of all vowels.

Hence:

$$\begin{aligned} A' &= \text{set of all letters which are not in A} \\ &= \text{set of all letters which are not vowels} \\ &= \text{set of all consonants} \end{aligned}$$

Now, what is $(A')'$? We have

$$\begin{aligned} (A')' &= \text{set of all letters which are not in A} \\ &= \text{set of all letters that are not consonants} \\ &= \text{set of all vowels} \\ &= A \end{aligned}$$

Thus $(A')' = A$

Example 1.24: Let $U = \{1, 2, 3, 4, 5, 6, 7\}$

$A = \{3, 4,\}$ and $B = \{4, 5,6\}$.

Then $A' = \{1, 2, 5, 6, 7\}$; $B' = \{1, 2, 3, 7\}$

What is $A' \cap B'$? Clearly

$A' \cap B' = \{1, 2, 7\}$

What is $A \cup B$? It is easily seen that

$A \cup B = \{3, 4, 5, 6\}$.

What is $(A \cup B)'$?

$(A \cup B)' = \{1, 2, 7\}$ Hence, we see that

$(A \cup B)' = \{1, 2, 7\} = A' \cap B'$

It can be shown that the above result is true in general. If A and B are any two subsets of the universal set U then

$$(A \cup B)' = A' \cap B'$$

Similarly,

$$(A \cap B)' = A' \cup B'$$

These two results are stated in words as follows:

The complement of the union of two sets is the intersection of their complements and the complement of the intersection of two sets is the union of their complements

Exercises 1.4

1 If $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, find the complements of the following sets:

(i) $A = \{2, 4, 6, 8\}$, (ii) $B = \{1, 3, 5, 7, 9\}$

(iii) $C = \{2, 3, 5, 7\}$ (iv) $\varnothing$,

(v) U

2. If U is the set of all natural numbers and A' is the set of all composite numbers, what is A?

3. For the set A and C of Exercise 1, verify that
$$(A \cup C)' = A' \cap C', \text{ and}$$
$$(A \cap C)' = A' \cup C'$$
4. Which of the following statements are true and which are false?
   (*i*) $U' = \varnothing$, (*ii*) $\varnothing' = U$
   (*iii*) For any two subsets, X and Y of U,
   $$(X \cup Y)' = X' \cup Y'$$
   (*iv*) For any two subsets X and Y of U
   $$(X \cap Y)' = X' \cap Y'$$
   (*v*) For any two subsets S and T of U,
   $$(S \cup T)' = S' \cap T'$$
   (*vi*) For any two subsets S and T of U,
   $$(S \cap T)' = S' \cap T'$$
5. Let U be the set of all triangles in a plane. If A is the set of all triangles with at least one angle different from 60^0, what is A'?

1.5 Venn Diagrams

Most of the ideas about sets and their properties can be visualised by means of diagrams which are known as "Venn diagrams." (Venn diagrams are named after the English logician, John Venn, 1834-1883). Usually, the universal set is represented by a rectangle and its subsets by circles, ellipses, etc.

If it is necessary, we specifically mark the elements of the set inside the diagram

Example 1.25

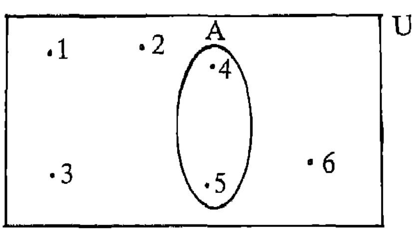


Fig 1 1

In the above diagram:
$$U = \{1, 2, 3, 4, 5, 6\},$$
$$A = \{4, 5\}.$$

Example 1.26

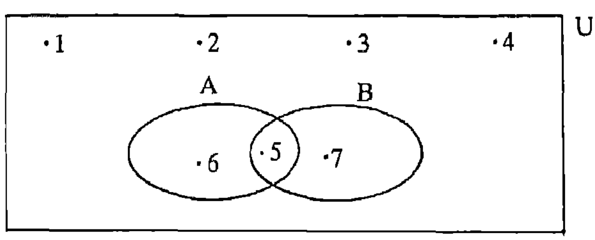


Fig 1 2

In the above diagram.
$$U = \{1, 2, 3, 4, 5, 6, 7\},$$
$$A = \{5, 6\},$$
$$B = \{5, 7\}.$$

Example 1.27

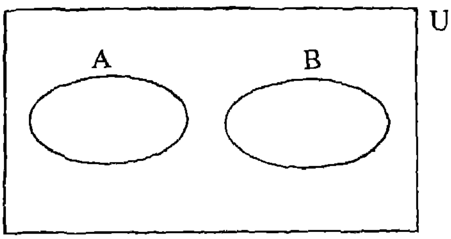


Fig 1.3

In the above diagram, A and B are disjoint.

Example 1.28

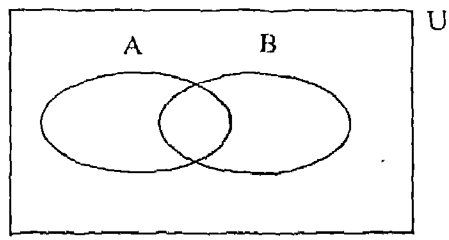


Fig 1 4

In the above diagram, the shaded portion is $A \cap B$

Example 1.29

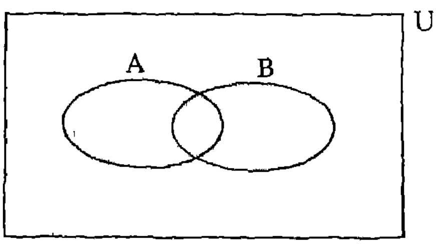


Fig 1.5

In the above diagram, the shaded portion is $A \cup B$.

Example 1.30

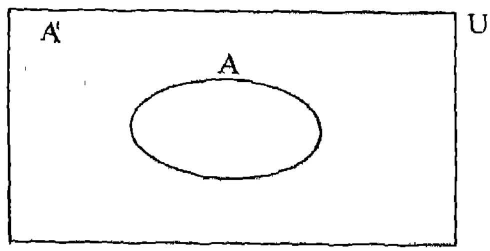


Fig. 1 6

In the above diagram, the shaded portion is A'.

Exercises 1.5

1. If $U = \{a, e, i, o, u\}$ and $A = \{a, i, o\}$, represents these sets in a Venn Diagram
2. Represents the following sets in a Venn diagram:

   (i) $U = \{2, 3, 5, 7, 11\}$, $A = \{2,3\}$

   (ii) $U = \{x : x \text{ is a natural number and } 2 \leq x \leq 8\}$

   $A = \{x . x \in U \text{ and } x \text{ divides } 18.\}$

   $B = \{x : x \in U \text{ and } x \text{ is a prime divisor of } 18.\}$

3. If U is the universal set, A and B are subsets of U such that $B \subset A$, represent these sets in a Venn diagram
4 Draw a Venn diagram for the following:
   (i) $A \cap B$ when $B \subset A$
   (ii) $A \cup B$ when $B \subset A$
5. If A, B and C are three subsets of the universal set U, draw a Venn diagram showing
   (i) $A \cup (B \cup C)$,
   (ii) $(A \cap B) \cap C$,
   (iii) $(A \cup B) \cup C)'$,
   (iv) $(A' \cap B)' \cap C'$,
   (v) From (iii) and (iv) can you conclude that $(A \cup B) \cup C)' = (A' \cap B)' \cap C'$?
6 If A, B, and C are three subsets of the universal set U, draw Venn diagrams for the following:
   (i) $B \cap C$, when $B \subset C$;
   (ii) A and C are disjoint sets and both A and C are subsets of B.

1.6 Applications

Consider the universal set

$U = \{1, 2, 3, 4, 5, 6, 7\}$ and the two subsets
$A = \{3, 5, 7\}$ and
$B = \{1, 2, 5, 6\}$.

If a set S has only a finite number of elements, we denote by $n(S)$ the number of elements of S. Thus in the above sets

$n(U) = 7$,
$n(A) = 3$,
$n(B) = 4$,

What is $A \cup B$? $A \cup B = \{1, 2, 3, 5, 6, 7\}$
What is $n(A \cup B)$? $n(A \cup B) = 6$
What is $n(A \cap B$? Since $A \cap B = \{5\}$, $n(A \cap B) = 1$
Now $n(A \cup B) + n(A \cap B) = 6 + 1 = 7$
Also $n(A) + n(B) = 3 + 4 = 7$
Thus, in this case we have

$$n(A \cup B) + n(A \cap B) = n(A) + n(B),$$
$$\text{i.e.}, n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

This result is true in general. For any two sets A and B, with finite number of elements, we have the following formula

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Try to give a proof of the above formula using Venn diagrams.

Example 1.31: Let us verify the above formula

Let $A = \{a, b, c, d, e\}$
$B = \{a, e, i, o, u\}$

$A \cup B = \{a, b, c, d, e, i, o, u\}$

$\therefore n(A \cup B) = 8$

$A \cap B = \{a, e,\},$

$\therefore n(A \cap B) = 2$

Further, $n(A) = 5, n(B) = 5$

Now $n(A) + n(B) - n(A \cap B) = 5 + 5 - 2 = 8.$

$\therefore n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Example 1.32: If A and B are two sets such that $A \cup B$ has 18 elements, A has 8 elements, and B has 15 elements, how many elements does $A \cap B$ have?

Solution: We have

$$n(A \cup B) = 18$$
$$n(A) = 8$$
$$n(B) = 15$$

We have the formula

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

So we have $18 = 8 + 15 - n\,(A \cap B)$

$\therefore n\,(A \cap B) = 23 - 18 = 5$

$\therefore n\,(A \cap B) = 5$

Second Method: We can also solve the above example as follows.

We want to find $n(A \cap B)$. Let $n(A \cap B) = x$, and in Fig. 1 7, mark x in the portion corresponding to $A \cap B$. Now $n(A) = 8$. Out of these 8 elements, x have already been marked The remaining $8 - x$ we now mark in A outside the $A \cap B$ portion. Also $n\,(B) = 15$. Again, x elements have already been marked. The remaining $15 - x$ elements we mark in B outside the $A \cap B$ portion. Now, from the above diagram we get

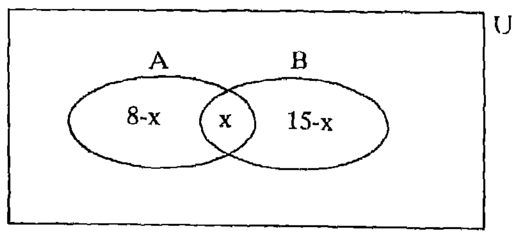


Fig. 1.7

$$n(A \cup B) = (8 - x) + x + (15 - x)$$
$$= 23 - x.$$

But we are given that $n\,(A \cup B) = 18$

$\therefore 18 = 23 - x$

$\therefore 18 = 23 - x$

Hence, $n\,(A \cap B) = x = 5.$

Example 1.33: In a class of 30 students, twenty students like to play cricket, and 15 like to play football. Also, each student likes to play at least one of the two games. How many students like to play both cricket and football?

Solution: We have the formula: $n(A \cup B) = n(A) + n(B) - n(A \cap B).$

Let A be the set of students who like to play cricket Let B be the set of students

who like to play football Then $A \cup B$ is the set of students who like to play at least one game; and $A \cap B$ is the set of students who like to play both games.

We have $n(A) = 20, n(B) = 15; n(A \cup B) = 30$.

From the formula we get

$$30 = 20 + 15 - n(A \cap B)$$

It is now easily seen that

$$n(A \cap B) = 5$$

$\because$ 5 students like to play both games. We give below the Venn diagram for the above problem:

$(20 - x) + x + (15 - x) = 30,$

So, $35 - x = 30$, and thus $x = 5$

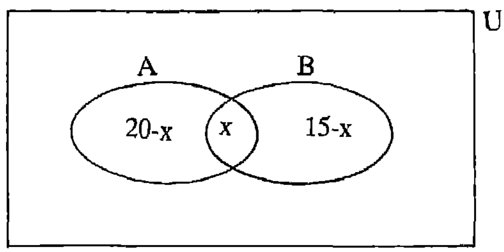


Fig. 1.8

Example 1.34: In a group of 50 people, 35 speak Hindi, 25 speak both English and Hindi, and all the people speak at least one of the two languages. How many people speak only English and not Hindi?

Solution: Let H denote the set of people speaking Hindi; E the set of people speaking English. We draw the Venn diagram

We first mark 25 in the common portion because 25 speak both Hindi and English. Then, since 35 speak Hindi and of these 25 have already been marked 10 is marked in the remaining portion in H. Now we do not know how many speak English only, and so we mark x there. We get

$$n(H \cup E) = 10 + 25 + x$$
$$= 35 + x$$

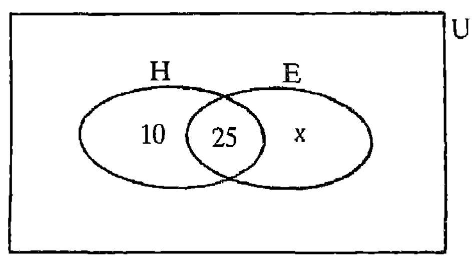


Fig. 1.9

But we are given $n(H \cup E) = 50$

$$\because 50 = 35 + x$$
$$\therefore x = 50 - 35 = 15$$

Thus 15 people speak only English and not Hindi. Since $x = 15$, we get from the Venn diagram

$$n(E) = 25 + x = 25 + 15 = 40$$

Hence 40 people speak English.

Exercises 1.6

1 If A and B are two sets such that $n(A) = 17$, $n(B) = 23$, $n(A \cup B) = 38$, find $n(A \cap B)$
2. If A and B are two sets such that A has 12 elements, B has 17 elements, and $A \cup B$ has 21 elements, how many elements does $A \cap B$ have?
3. If S and T are two sets such that S has 21 elements, T has 32 elements, and $S \cap T$ has 11 elements, how many elements does $S \cup T$ have?
4 If X and Y are two sets such that X has 40 elements, $X \cup Y$ has 60 elements and $X \cap Y$ has 10 elements, how many elements does Y have?
5. If A and B are disjoint sets, show that
$$n(A \cup B) = n(A) + n(B)$$
6. In a group of 70 people, 37 like coffee, 52 like tea and each person likes at least one of the two drinks. How many people like both coffee and tea?
7 In a group of 65 people, 40 like cricket, 10 like both cricket and tennis How many like tennis only and not cricket? How many like tennis?
8. In a committee 50 people speak French, 20 speak Spanish and 10 speak both Spanish and French. How many speak at least one of these two languages?

CHAPTER 2

Real Numbers Functions and Graphs

2.1 Rational Numbers

You are familiar with the numbers 1, 2, 3, . . . , which are used in counting objects. Hence, these numbers are called *counting numbers* They are also called *natural numbers*. The set of all natural numbers is denoted by "N." Thus,

$$N = \{1, 2, 3, \ldots\}$$

The three dots after the number 3 stand for the natural numbers which come after 3, namely, the numbers 4, 5, 6, and so on You know that two natural numbers can be added, and their sum is again a natural number This property of natural numbers is described by saying that *the set of natural numbers is closed with respect to the operation of addition* As regards the operation of subtraction, this property does not hold good Subtracting one natural number from another does not always lead to a natural number. For example, if you subtract 8 from 5 you do not get a natural number. This is so because there is no natural number which can be added to 8 to get 5 The introduction of negative integers $-1, -2, -3, \ldots$ and the number 0 (zero) enabled us to remedy this defect. The negative integers, the number zero, and the natural numbers (also called positive integers), taken together form a set which is called "the set of integers," and is denoted by "Z" Thus,

$$Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

The sum of any two integers (positive, negative or zero) is always an integer, and if you subtract an integer from any other integer, the result is always an integer. Thus we see that the set Z *is closed with respect to the operations of addition and subtraction*

The set Z of all integers has another property. If one integer (positive, negative or zero), is multiplied by another integer (positive, negative or zero), the product is again an integer. Thus the set Z is closed with respect to the operation of multiplication. But Z is not closed with respect to the operation of division, which is the inverse of multiplication. For example, there is no integer by which you can multiply 3 to get 2. Such a situation necessitates the introduction of *new numbers*. A new number, denoted by the symbol $\frac{2}{3}$, is introduced with the property that the new number $\frac{2}{3}$ multiplied by 3, gives 2 $\frac{2}{3}$ is not an integer and is called a *rational number*, which is read as "2 divided by 3," "2 by 3" or "2 upon 3."

In general, if p is any integer and q is a non-zero integer, then the symbol $\frac{p}{q}$ represents

a rational number, which has the property that its product with q is the integer p. In this manner with every pair of integers p and $q, q \neq 0$, a rational number $\frac{p}{q}$ can be associated The condition that $q \neq 0$, is necessary, as division by zero is not defined Thus the introduction of rational numbers makes it possible to perform the operation of division.

The set of all rational numbers is denoted by Q Thus,

$$Q = \{\frac{p}{q} \;\; p \text{ and } q \in Z \text{ and } q \neq 0\}$$

When we write a rational number in the form $\frac{p}{q}$, there is no loss of generality in taking $q > 0$, whereas p may be positive, negative or zero We will follow this convention.

Every integer (positive, negative or zero) can be written in the form $\frac{p}{q}$, where $q = 1$ For example, $3 = \frac{3}{1}$, $-5 = \frac{-5}{1}$ or $0 = \frac{0}{1}$ and so forth Hence it is clear that every integer is a rational number and that the set Z of all integers is a subset of the set Q of all rational numbers In symbols, $Z \subset Q$

Just as, corresponding to every positive integer n there is a negative integer $-n$, similarly, corresponding to every positive rational number $\frac{p}{q}$ there exists a negative rational number $\frac{-p}{q}$

In a lower class you learnt the methods of adding and multiplying two rational numbers. You also learnt as to how to subtract a rational number from another and how to divide one rational number by another non-zero rational number. These operations, as defined for rational numbers, are quite consistent with the corresponding operations defined for integers This means that performing an operation with two integers, treating them as rational numbers, will give the same result, as is given by performing the corresponding operation with them as defined for integers

Two rational numbers $\frac{p}{q}$ and $\frac{r}{s}$ are said to be equal to each other if and only if $ps = rq$ Stated differently

$$\frac{p}{q} = \frac{r}{s}, \text{ if and only if } ps - rq = 0$$

Also,

$$\frac{p}{q} > \frac{r}{s}, \text{ if and only if } ps - rq > 0$$

and

$$\frac{p}{q} < \frac{r}{s}, \text{ if and only if } ps - rq > 0$$

If $p = km$ and $q = kn$, where k and n are positive integers, and p, m are either both

positive or both negative, then $\frac{p}{q} = \frac{m}{n}$. For $p\,n = k\,m.n = m.k\,n = m\,q$ Hence $\frac{p}{q} = \frac{m}{n}$
For example, $\frac{4}{6} = \frac{2}{3}$, since $4 = 2 \times 2$ and $6 = 2 \times 3$

It follows from the above discussion that the set Q of rational numbers is closed with respect to the four fundamental operations of addition, subtraction, multiplication and division with the provison that division by zero is not defined It is also now clear that given two rational numbers a and b, only one of the following three possibilities will hold good:

either $a = b$, or $a > b$, or $a < b$.

Stated differently, if a and b *are two distinct* (non-equal) *rational numbers then one of them is greater than the other* The relation 'greater than' is called **Order Relation** This order relation has the following properties:

1. If a and b are rational numbers, then only one of the following three possibilities will hold good:
   $(i)\ a > b$, $(ii)\ a = b$, $(iii)\ b > a$ (We have already noticed this property)
2. If $a > b$ and $b > c$, then $a > c$, for a, b, c in Q
3. If $a > b$, then $a + c > b + c$, for a, b, c in Q
4. If $a > b$, $c > 0$, then $a\,c > b\,c$, for a, b, c in Q

The order relation defined in the set of rational numbers will be more clearly understood when we consider their representation on the *number line* in the next section

2.2 The Number Line

We will now consider the representation of rational numbers on a straight line, which extends endlessly in both the directions, as shown in the figure below, Fig. 2 1

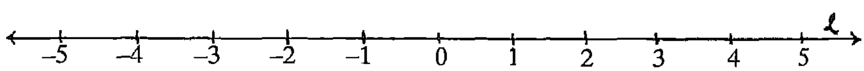


Fig 2.1

The straight line l extends in both the directions endlessly as indicated by the arrowheads. Choose a point on l and label it 0 (zero) Next, choose another point on the line l to the right of 0 and label it 1 (one). We agree that the point 0 represents the number zero and the point labelled 1 represents the number 1. The length of the line segment between the points 0 and 1 is the unit length for our purpose Again, mark on l, to the right of the point 1, points labelled 2, 3, 4, 5, .. in such a manner *that the line segment* between any two consecutive points, is of unit length. The points labelled 2, 3, 4, represent respectively the numbers 2, 3, 4. Since the line l extends endlessly on the right of the point 0, to every positive integer there will correspond a point on l.

Similarly, mark points -1, -2, -3, and so on the line l to the left of the point 0 In this way to every negative integer there will correspond a point on l to the left of 0, for the line l extends endlessly to the left of 0.

The point on l, representing the number 1, has been labelled 1, and the point representing the number 2, has been labelled 2, and this is the case with every integer There is a good advantage in this manner of labelling points on l. This enables us to identify the number with the point which represents the number We can now speak of the points as the numbers 1, 2, 3, . . The point 0 is called the *origin*, and the half line to the right of 0 is called the *positive half line*, since the positive numbers are located on it The half line to the left of 0 is called the *negative half line*

Addition on the Number Line

The numbers on the number line l are located from left to right in increasing order. This means that of two numbers the greater one is on the right of the smaller one For example, since 4 is greater than 3, the point labelled 4 is on the right of the point labelled 3 Similarly, -1 is on the right of -3, since $-1 > -3$ Another advantage of the number line is that the results of adding two integers and of subtracting one integer from another can be found at a glance For example, in adding 3 to 2 we have just to count 3 unit segments to the right of 2 and we reach 5 which is the answer. But if we have to add a negative integer, (or subtract a positive integer), we have to count as many unit segments to the left For example, if -3 is to be added to 4, then we have to count 3 unit segments to the left of 4 to get 1 Adding a positive integer or subtracting a negative integer means counting as many steps to the right. For example, if -3 is to be subtracted from -1, then we have to count 3 unit segments to the right of -1. This gives 2.

Representation of Rational Numbers

Next, we consider the representation of the rational numbers on the number line l. The midpoint of the segment between 0 and 1 represents the number $\frac{1}{2}$ Mark on l to the right of the point $\frac{1}{2}$ new points such that the length of the line segment between two consecutive new points has half the unit length These points will successively represent the numbers $\frac{2}{2}, \frac{3}{2}, \frac{4}{2}, \frac{5}{2}, \ldots$ Thus all those positive rational numbers which have 2 as their denominators will be represented by points on l Repeating the same process to left of 0, we get all negative rational numbers which have 2 as their denominators, e.g. the numbers $-\frac{1}{2}, -\frac{2}{2}, -\frac{3}{2}, \ldots$ Next, taking one-third of the unit length and marking points on l to the right of 0 such that the length of the line segment between any two successive points is one-third of the unit length, we get points on l which respectively represent rational numbers $\frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}, \frac{5}{3}, \ldots$ Thus all the positive rational numbers having 3 as the denominator will be represented on l. In the same way we mark points

on the left of 0, to represent negative rational numbers $-\frac{1}{3}, -\frac{2}{3}, -\frac{3}{3}, -\frac{4}{3}, -\frac{5}{3}, \ldots$ Similarly, we mark points on l to the right of 0 corresponding to positive rational numbers having denominators 4, 5, 6, 7, 8, . . , and repeating the same process to the left of 0, we get points representing the corresponding negative rational numbers In this way all the rational numbers will be represented by points on the number line l. Some of the rational numbers are represented by points on l in the following figure

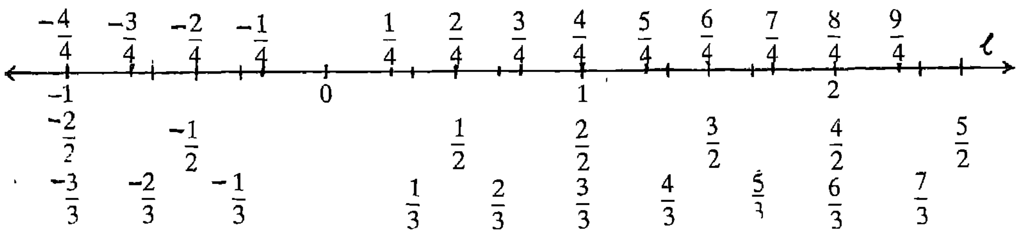


Fig. 2 2

It will be noticed that certain points on the line l appear to represent more than one rational number. For example, the point, representing the positive integer 2, also represents the numbers $\frac{4}{2}, \frac{6}{3}, \frac{8}{4}$ and so on There is nothing unnatural about it, since $\frac{4}{2} = \frac{6}{3} = \frac{8}{4} = 2$. Hence there is no ambiguity about this representation. Thus every rational number is represented by one and only one point on the number line. A question naturally arises here. Does every point on the number line represent a rational number? The answer to this question is in the negative, and we will give in Section 2.4 the justification of this answer.

2.3 Non-rational Numbers

We first prove that *there is no rational number whose square is 2.*

Proof: Since $1^2 = 1$ and $2^2 = 4$, it follows that if 2 is the square of a positive rational number, it cannot be an integer and it must be greater than 1 Suppose that there is a rational number $\frac{p}{q}$ such that its square is 2. Without any loss of generality we can suppose that the integer q is greater than 1 and that p and q have no common factors. Then

$$2 = \frac{p^2}{q^2}$$

Multiplying both the sides of this equality by q we get

$$2q = \frac{p^2}{q}$$

Now $2q$ is clearly an integer. On the other hand, p^2 and q have no common factor, as p and q have no common factor, so that $\frac{p^2}{q}$ is a fraction different from an integer, for $q > 1$. Hence $2q \neq \frac{p^2}{q}$ This contradiction proves that our assumption is false and that 2 is not the square of a rational number

The reasoning, followed in proving that 2 is not the square of a rational number, can be applied to show that the numbers 3, 5, 6 and 7 are not squares of rational numbers In general, it can be shown that a positive integer m, which is not a perfect square (i.e , which is not the square of a positive integer), is not the square of a rational number The proof of this general statement is similar to the proof of the fact that 2 is not the square of a rational number.

2.4 Inadequacy of Rational Numbers

We use rational numbers very often in everyday life When we measure lengths, or distances, or weights, we use rational numbers, e g , $2\frac{1}{2}$ metres, $5\frac{3}{4}$ kilometres or $3\frac{1}{4}$ kilograms As we will now show, there are lengths which cannot be measured in terms of rational numbers. Before we do that let us recall a result from geometry, the so called Pythagoras theorem.

The square of the hypotenuse of a right angled triangle is equal to the sum of the squares of the other two sides.

Let ABC be a right-angled triangle, right-angled at B, such that $AB = BC = 1$ unit. Suppose that $AC = x$ units Then by the Pythegoras theorem

$AC^2 = AB^2 + BC^2$

i.e. $x^2 = 1 + 1 = 2$.

Since $x^2 = 2$, x cannot be a rational number, as we saw earlier in Section 2 3 Thus the length of AC cannot be measured in terms of rational numbers.

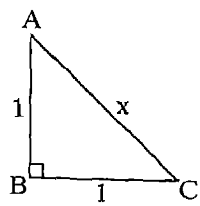


Fig 2.3 (a)

This shows the inadequacy of rational numbers in measuring lengths. however, the segment AC has a finite length and we have to express it in numbers using the unit of length. It can be said that there is a non-rational number whose square is 2 and therefore, this number is denoted by $\sqrt{2}$ enabling us to say that the length of the hypotenuse is $\sqrt{2}$ units. Instead of using the term "non-rational," we say that $\sqrt{2}$ is an *irrational number*. Similarly, we define $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$ and say that they are irrational numbers.

Let us try to represent the number $\sqrt{2}$ on the number line l. Denote the point 0 on the line l by A, and the point 1 by B. Then the line segment AB is of unit length At the point B draw a straight line perpendicular to the line l and cut off a segment BC of unit

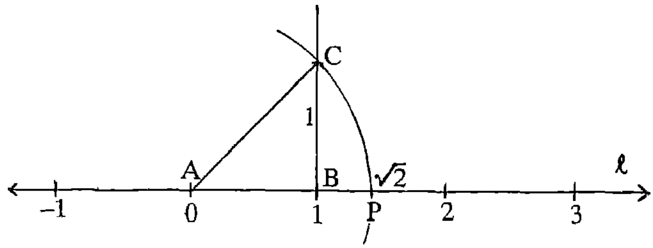


Fig. 2 3(b)

length from this perpendicular line. With A as centre, and AC as radius draw a circular arc. Let this arc intersect the line l in p as shown in the figure 2 3 (b). Then $AP = AC$. Since ΔABC is a right-angled triangle, with its right angle at B, and $AB = BC = 1$, it follows from the Pythagoras theorem that $AC = \sqrt{2}$ Hence AP $= \sqrt{2}$.

Thus the point P on the number line corresponds to the irrational number $\sqrt{2}$ We have now discovered a point on the number line which does not represent any rational number. Geometrical constructions can be devised to identify the points on the number line, which correspond to the irrational numbers $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$ and so on. It may be pointed out here that irrational numbers are not obtained only by the method of root extraction as is the case with $\sqrt{2}$, $\sqrt{3}$, and so forth We shall discuss later a general method of identifying irrational numbers It may, however, be mentioned here that there are infinitely many irrational numbers, and in a sense, there are "more" irrational numbers than rational numbers

It will be observed that in the process of representing rational numbers by points on the line l, we were free to choose two points, which were labelled 0 and 1 The rest of the points were determined with the help of the unit length, which was known once the points 0 and 1 were identified. This means that all the rational numbers can be obtained, starting from the number 1, by using the four fundamental operations of arithmetic. As regards the irrational numbers, we have to go beyond the four fundamental operations

2.5 Properties of Rational Numbers

In the set Q of all rational numbers there are two basic operations, namely, the operation of *addition* (+) and the operation of *multiplication* (.) Subtraction and division are respectively inverse operations of addition and multiplication. You know that the sum of two rational numbers is a rational number and the product of two rational numbers is also a rational number This means that the set Q is closed with respect to the two operations (+) and ( ). There are certain familiar properties. satisfied by Q in relation to the two operations, which we list below In what follows we

use the letters a, b, c for rational numbers.

1 $a + b = b + a$ for all $a, b \in Q$ (Commutative law for addition)

2. $(a + b) + c = a + (b + c)$ for all $a, b, c \in Q$ (Associative law for addition)

3. The rational number 0 is such that
$a + 0 = 0 + a = a$ for all $a \in Q$ (Additive identity)

4. To each $a \in Q$, there is a number $-a \in Q$ such that $a + (-a) = (-a) + a = 0$ (Additive inverse)

5 $a.b = b.a$ for all $a, b \in Q$ (Commutative law for multiplication)

6 $(a.b).c = a\,(b\,c)$ for all a, b, c $\in Q$ (Associative law for multiplication)

7. The rational number 1 (unity), is such that
$1\,a = a\,1 = a$ for all $a \in Q$, (Multiplicative identity)

8. To every non-zero $a \in Q$ there corresponds a rational number $\frac{1}{a}$ such that $a.\frac{1}{a} = \frac{1}{a}\,a = 1$, (Multiplicative inverse)

9. For $a, b, c \in Q$
$$\left.\begin{array}{l} a\,(b + c) = a.b + a\,c \text{ and} \\ (a + b).c = a\,c + b.c \end{array}\right\}$$ (Distributive law)

The system { Q, + , } is said to be a **Field** because of the nine properties listed above. Simply speaking, the rational numbers under the usual operations of + and . form a *field* It is called the field of rational numbers

You have been using all the nine properties listed above all the while. They have been listed here because later, when you come across systems, other than the rational numbers, satisfying the above properties, you may recognize them as *fields*

2.6 Decimal Representation of Rational Numbers

You know that every rational number can be represented either as a terminating decimal or as a non-terminating repeating decimal. For example, $\frac{1}{2} = 0.5$, $\frac{7}{5} = 1\,4$, and $\frac{1}{3} = 0.333\ldots$, $\frac{7}{6} = 1.6666$. In the decimal representation of $\frac{1}{3}$, the digit 3 goes on repeating, and in the representation of $\frac{7}{6}$, the first digit before the decimal point is 1, and after 1 the digit 6 goes on repeating. You are perhaps familiar with the following result. If p and q are positive integers, having no common factors the rational number $\frac{p}{q}$ will have a terminating decimal only when the prime factors of q are only twos and fives, i.e. $q = 2^m \times 5^n, m, n = 0, 1, 2, 3, \ldots$ For example, $\frac{7}{8}, \frac{3}{20}, \frac{11}{25}$ and $\frac{99}{200}$ will all have terminating decimal representations. (The proof is not difficult and should

be given in the class room by the teacher concerned) Repeating decimals are also called 'periodic decimals' or 'recurring decimals'. Repeating decimals which consist of only one repeating digit, are written simply by putting a dot( ) above the repeating digit, e g , $\frac{7}{6}$ = 1.6666 . is written as $1\dot{6}$ If, however, the number of digits in the repeating part is more than one, then a dot is put on the first digit and another on the last digit of the repeating part, e.g., $\frac{16}{7}$ = 2.142857142857 .., which is simply written as $2.\dot{1}4285\dot{7}$, meaning thereby that the entire block of six digits 142857 is repeating. Sometimes a line, called *vinculum*, is drawn covering the entire block of repeating digits, e g $\frac{15}{7}$ = $2.\overline{142857}$

Conversely, every terminating decimal and every repeating decimal can be converted into a rational number of the form $\frac{p}{q}$, where p and q are integers, $q > 0$. For example, $0.25 = \frac{25}{100} = \frac{1}{4}$ A different method is needed to convert a repeating decimal into a fraction $\frac{p}{q}$. Take the decimal 0 333 Let x = .333 . Multiplying both the sides by 10 we get $10x$ = 3 333 = $3 + x$

$$\therefore 10x - x = 3$$

$$\text{i.e. } 9x = 3, \text{ and } x = \frac{3}{9} = \frac{1}{3}$$

This method, however, needs justification, which cannot be given here But the method is correct.

A decimal with a repeating 9 can be converted into a terminating decimal by increasing the last digit before 9 by one. A terminating decimal can be converted into a repeating decimal either by adding a repeating zero to the right of the last digit after the decimal point, or by reducing the last digit by 1 and adding a repeating 9. For example, 0.1999 . = 0.2

and 0 1 = 0.100000 . . .

or 0.1 = 0 09999 . .

It is, thus, clear that the same number can have three types of decimal representations. In order to make the decimal representation unique we adopt the convention that a terminating decimal will be represented as a decimal with a repeating zero and a decimal with a repeating 9 will be converted into a decimal with a repeating zero This convention enables us to assert that every rational number has a unique (non-terminating) repeating decimal representation and conversely, every repeating decimal represents a rational number.

Example 2.1: Between any two distinct rational numbers, a and b, there exists another rational number.

Proof: Since a and b are distinct, there is no loss of generality in supposing that $a < b$. The number $\frac{a+b}{2}$ is rational, and $a < \frac{a+b}{2} < b$.

For, $\frac{a+b}{2} > \frac{a+a}{2} = a$, since $b > a$

Again, $\frac{a+b}{2} < \frac{b+b}{2} = b$, since $b > a$

Thus, $a < \frac{a+b}{2} < b$

Second Proof: b is greater than a by $b - a$

$$\therefore a < a + \frac{(b-a)}{2} < b,$$

$$\text{i.e., } a < \frac{2a+b-a}{2} < b,$$

$$\text{Hence, } a < \frac{a+b}{2} < b$$

For inserting n rational numbers between a and b, divide $(b - a)$ by $(n + 1)$ and the required rational numbers will be $a + \frac{b-a}{n+1}$, $a + \frac{2(b-a)}{n+1}$, $a + \frac{3(b-a)}{n+1}$, ...

. . , $a + \frac{n(b-a)}{n+1}$

The proof is simple and is left to the reader

Corollary: It follows from this result that between two distinct rational numbers there are infinitely many rational numbers

Example 2.2: Prove that 7 is not the cube of a rational number

Proof: Suppose, to the contrary, that there is a rational number $\frac{p}{q}$ such that

$$\left(\frac{p}{q}\right)^3 = 7.$$

Since $1^3 = 1$, and $2^3 = 8$, it follows that $1 < \frac{p}{q} < 2$

Then $q > 1$, for if $q = 1$, $\frac{p}{q}$ will be an integer, and there is no integer between 1 and 2.

We can also suppose that p and q have no common factors, because if they had any common factor, that will cancel out and the new numerator and denominator will have no common factor

$$\text{Now, } 7 = \frac{p^3}{q^3}$$

Multiplying both the sides by q^2 we get

$$7q^2 = \frac{p^3}{q}$$

q being an integer $7q^2$ is an integer, and since $q > 1$ and q does not have a common factor with p and consequently with p^3 So, $\frac{p^3}{q}$ is a fraction different from an integer. Thus $7q^2 \neq \frac{p^3}{q}$. This contradiction proves the result

Exercises 2.1

1. Represent the following rational numbers on the number line:

   (*i*) $\frac{8}{3}$ (*ii*) $\frac{15}{7}$

   (*iii*) $\frac{7}{6}$ (*iv*) 1.3

   (*v*) 2.51 (*vi*) 3.13

2. Give three rational numbers lying between $\frac{1}{3}$ and $\frac{1}{2}$

3. Give four rational numbers lying between $\frac{1}{4}$ and $\frac{1}{3}$

4. Find three rational numbers lying between 0 and 0.1 Find twenty rational numbers between 0 and 0.1. Give a method to determine any number of rational numbers between 0 and 0.1.

5. Find three rational numbers lying between $-\frac{2}{5}, -\frac{1}{5}$.

6. Which of the following rational numbers have the terminating decimal representation?

   (*i*) $\frac{3}{5}$ (*ii*) $\frac{7}{20}$

   (*iii*) $\frac{2}{13}$ (*iv*) $\frac{27}{40}$

   (*v*) $\frac{13}{125}$ (*vi*) $\frac{23}{7}$

   Hint: Use the result that a rational number $\frac{p}{q}$ where p and q have no common factor(s) will have a terminating representation if and only if the prime factors of q *are* 2's or 5's or both (In $\frac{3}{40}$, $40 = 2^3 \times 5$ and in $\frac{7}{16}$, $16 = 2^4$)

7. Find the decimal representation of $\frac{1}{7}, \frac{2}{7}$ Deduce from the decimal representation of $\frac{1}{7}$, without actual calculation, the decimal representation of $\frac{3}{7}, \frac{4}{7}, \frac{5}{7}$ and $\frac{6}{7}$.

8. If a and b are two rational numbers, prove that $a+b, a-b, ab$ are rational numbers. If $b \neq 0$, show that $\frac{a}{b}$ is also a rational number

9. You have seen that $\sqrt{2}$ is not a rational number Show that $2+\sqrt{2}$ is not a rational number.

10 Prove that $3\sqrt{3}$ is not a rational number.
11 Show that $\sqrt[3]{6}$, $\sqrt[4]{3}$, $\sqrt[5]{5}$ are not rational numbers
12. If a is a positive rational number and n is a positive integer greater than 1, prove that a^n is a rational number.
13 Let m, n be positive integers such that $m \geq 1$, $n \geq 1$ and m is not a perfect nth power, i e., there is no positive integer p such that $p^n = m$. Prove that there is no rational number a such that $a^n = m$.

2.7 Irrational Numbers

As we remarked in Section 2.4, irrational numbers are not obtained only by extracting square roots or cube roots of positive integers which are not perfect squares or perfect cubes As we will see below, irrational numbers are represented by a special type of decimals. For the sake of simplicity, we consider here numbers which lie between 0 and 1

According to a convention adopted by us in Section 2 6, every rational number is represented by a non-terminating repeating decimal, and conversely every non-terminating repeating decimal represents a rational number This representation is *unique*. Now consider the following decimal expression

$$0.101001000100001\ldots \qquad (1)$$

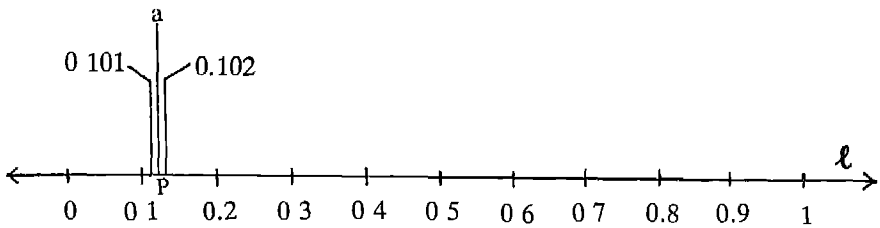
Observe that in the above decimal expression (1), on the right of the decimal point there are either 1's or zeros, and that the 1's are separated respectively by one zero, then two zeros, then three zeros and so on. Thus the number of zeros separating two successive 1's goes on increasing by 1 successively. This shows that the decimal expression (1) is non-terminating and non-repeating Hence it cannot represent a rational number. We say that the decimal expression in (1), by definition, represents an "irrational number." We will see later that irrational numbers like $\sqrt{2}$, $\sqrt{3}$ also have non-terminating and non-repeating decimal representations

Denoting the irrational number, given in (1), provisionally by a, let us examine as to where this irrational number a stands in relation to rational numbers, and also see whether there is a point corresponding to a on the number line l. It is easily seen that

$0.1 < a < 0.2$, where 0.1 and 0.2 are rational numbers
Further, $0.101 < a < 0.102$.
Again, $0.101001 < a < 0.101002$,
and so on.

Continuing in this manner, we find closer and closer approximations of the irrational number a by rational numbers

Let us locate this irrational number a on the number line l



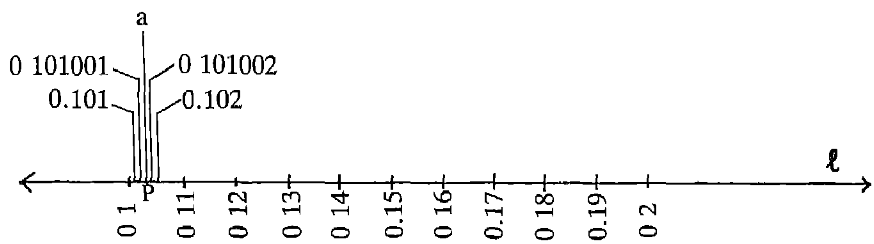


Fig. 2.4

First, we observe that a lies on the line segment whose end points are 0.1 and 0.2 Then, we observe that a lies on a smaller segment with end points 0 101 and 0 102. It is easily seen that this second segment is completely contained in the first segment. Again, a lies on a still smaller segment with end points 0.101001 and 0.101002. This third segment is completely contained in the second segment. Thus we see that there is a succession of small segments, each containing the number a and each segment containing the following one, in such a manner that the lengths of the segments go on decreasing successively. Hence we can intuitively see that this succession of segments will shrink into a point p on the line l. Since every segment contains a, this point p represents the irrational number a, given by the non- terminating and non-repeating decimal in (1). Some of the successive segments containing the irrational number a are shown in Figure 2.4. The preceding discussion has enabled us to arrive at the following conclusions·

1 An irrational number can be approximated, as closely as we like, by rational numbers,
2. To an irrational number, there corresponds a unique point on the number line l.

2.8 Decimal Representation of Irrational Numbers

We saw in an earlier section that $\sqrt{2}$ is an irrational number. Let us find out whether $\sqrt{2}$ has a decimal representation, and if so, examine the nature of the decimal representation The process of finding the square root of 2 by the division method can give us the decimal representation of $\sqrt{2}$. We will, however, follow here a more elementary method. It is easily seen that

$$1^2 = 1 < 2 < 4 = 2^2$$

Taking positive square roots we get

$$1 < \sqrt{2} < 2.$$

Next,

$$(1.4)^2 = 1.96 < 2 < 2.25 = (1.5)^2.$$

Taking positive square roots again, we have

$$1.4 < \sqrt{2} < 1.5$$

Further,

$$(1.41)^2 = 1.9881 < 2 < 2.0164 = (1.42)^2$$

Again, taking the positive square roots, we obtain

$$1.41 < \sqrt{2} < 1.42$$

If we continue this process the next step will lead to the following inequalities

$$1.414 < \sqrt{2} < 1.415$$

Proceeding in this manner, every new step will give us a closer decimal approximation of $\sqrt{2}$ than the previous step. The eighth step will give the following inequalities·

$$(1.4142135)^2 = 1.99999982358225 < 2 < 2.00000010642496 = (1.4142136)^2$$

Hence $1.4142135 < \sqrt{2} < 1.4142136$

This is a very close approximation of $\sqrt{2}$

Since $\sqrt{2}$ is not a rational number, this process will not terminate and will lead to a decimal expansion which will not terminate, nor will it be repeating. Hence the non-terminating and non-repeating decimal expansion of $\sqrt{2}$ will be given by

$$\sqrt{2} = 1.4142135\ldots \qquad (2)$$

Where the dots indicate that this decimal representation will not terminate Similarly, we can show that the decimal representations of $\sqrt{3}$, $\sqrt{5}$, $\sqrt[3]{7}$, and other non-rational numbers obtained by processes of root extractions, will not terminate nor will they be repeating Hence we conclude that

A number is irrational if and only if its decimal representation is non-terminating and non- repeating.

We can find numerous decimal expressions, like what is given in (1) of Section 2.7, by changing the digits and by varying the frequencies of a digit's occurrence in the decimal expression. For example, consider the following three decimal expressions.

(*i*) 0.12112111211112 . . .

(*ii*) 0.020020002 . .

(*iii*) 0.3000300003000003 .

All the three decimal expressions represent irrational numbers We can go on multiplying such examples endlessly Hence we conclude that

There are infinitely many irrational numbers. There is a point on the line l corresponding to every irrational number

We can locate these points on the line l as we did for the irrational number a, given by the decimal expression in (1) of Section 2 7. Thus we see that on the number line l there are infinitely many points which do not correspond to rational numbers

It was proved earlier that between two distinct rational numbers, howsoever close to each other they may be, there are infinitely many distinct rational numbers. This result gives the impression that rational numbers are very 'densely' located on the number line l Now we find that, even though the rational numbers are densely located on the line l, infinitely many points on l are left out and these points correspond to irrational numbers. *Rational numbers and irrational numbers taken together form the set of* **Real Numbers** This set is denoted by R. Thus every real number is either a rational number or an irrational number In either case, it has a non-terminating decimal representation. If this representation is repeating (including repeating zeros) it is a rational number, and if it is non-repeating it is an irrational number. It is clear that $Q \subset R$. Also, corresponding to every real number there is a unique point on the number line l It can also be shown that every point on the line l corresponds to a real number (rational or irrational). Let us now state the final conclusion:

To every real number there corresponds a unique point on the number line, and conversely, to every point on the number line there corresponds a real number

It may be noted that this correspondence it one-to-one, and for this reason the number line is called the '*real number line*'

2.9 Operations with Real Numbers

Now that we have extended the number concept from rational numbers to real numbers, it is natural to ask whether the fundamental operations of addition, subtraction, multiplication and division can be extended to the real numbers also. The answer is: "yes, the operations can be extended to the real numbers." In the first place we observe that the order relation 'greater than' holds in the case of real numbers also. Given two distinct real numbers, one of them is greater than the other. The greater real number lies to the right of the smaller one on the real number line. Every real number on the right of 0 on the number line is positive and every real number on the left of 0 is negative Corresponding to every positive real number a there is a negative real number $-a$. As regards the operation of addition, two real numbers can be added on the number line. When we add two rational numbers, we get the answer in a compact form. For example, $\frac{2}{3} + \frac{3}{4} = \frac{17}{12} = 1\frac{5}{12}$. On the other hand, although *the sum of two real numbers is a real number,* it is not always written in a simplified form. For example, the sum of 2 and $\sqrt{3}$ is just written as $2 + \sqrt{3}$; also the sum of $\sqrt{2}$ and $\sqrt{3}$ is just written as $\sqrt{2} + \sqrt{3}$. As regards multiplication, defining the product of two real numbers is

not easy on the number line. It is, however, true that *the product of two real numbers is always a real number.* The product of 2 with $\sqrt{3}$ is simply written as $2\sqrt{3}$, and the product of $\sqrt{2}$ and $\sqrt{3}$ is written as $\sqrt{2}.\sqrt{3}$ or $\sqrt{2.3} = \sqrt{6}$. We similarly deal with the operations of subtraction and division. Division by zero is not defined

Real numbers also possess all the properties of rational numbers listed in Section 2.5. You have only to replace there the word 'rational' by 'real' and the set Q by the set R. Thus the set R of real numbers is also a *field* under the operations of addition (+) and multiplication (.). Real numbers are also *ordered.* There is however, an important difference:

> *Every point of the number line corresponds to a real number. This is not true of rational numbers.*

2.10 Absolute Value of a Real Number

Let a be a real number (rational or irrational) Then there is a point on the number line l which corresponds to the number a. The distance of this point from the point 0 is called the ***absolute value*** of a. Since distance is always non-negative, the absolute value of a number (positive or negative) is always non-negative. The absolute value of a real number a is written as $| a |$.

Thus,

$$|2| = 2 \text{ and } |-2| = 2.$$

We can also say that

$$|a| = a \text{ if } a \geq 0,$$

and

$$|a| = -a \text{ if } a < 0.$$

Thus,

$$|3| = 3 \text{ since } 3 > 0.$$

and

$$|-3| = -(-3) = 3, \text{ since } -3 < 0.$$

The absolute value of 0 is 0, and the absolute value of a non-zero real number is always positive

Example 2.3: Find all the real numbers x on the number line which satisfy the inequality $|x| < 2$.

Solution: Suppose $x \geq 0$. Then the inequality $|x| < 2$ will reduce to $x < 2$ and $x \geq 0$.

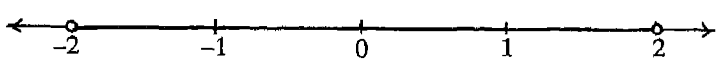


Fig. 2.5.

This means that all the points on the number line lying between 0 and 2, including 0 and excluding 2, will satisfy this part of the inequality.

Now, let $x < 0$. Then $|x| = -x$

If $-x < 2$, then $x > -2$ (An inequality is reversed by changing signs.)
Since $x < 0$, we have $-2 < x < 0$.
Hence all the points lying between -2 and 2, excluding the points -2 and 2, will satisfy the inequality $|x| < 2$

Example 2.4: (*i*) If a is irrational, then $-a$ is also irrational

(*ii*) The sum of a rational number with an irrational number is always irrational

Imp

(*iii*) The product of a non-zero rational number with an irrational number is always irrational.

Proof: (*i*) Let a be irrational. If $-a$ is not irrational, then $-a$ must be rational, since it is a real number. We know that the negative of a rational number is always rational, hence $-(-a)$ must be rational. But $-(-a) = a$, which is irrational. This contradiction proves that our supposition that $-a$ is not irrational is false. This proves the result.

(*ii*) Suppose that a is rational and b is irrational. Now, if $a + b$ is not irrational, then $a + b$ must be rational. If we subtract the rational number a from the rational number $(a + b)$, then the remainder must be rational. But $(a + b) - a = b$, which is irrational. This contradiction proves that $a + b$ must be irrational

(*iii*) Let a be a non-zero rational number, and b an irrational number. If $a.b$ is not irrational, $a\,b$ must be rational. Now, dividing the rational number $a\,b$ by the non-zero rational number a will give a rational number, but when we divide $a.b$ by a we get b which is supposed to be irrational. This contradiction proves the result

Example 2.5: The sum and product of two rational numbers are always rational, but neither the sum nor the product of two irrational numbers is always an irrational number.

Solution: We know that $\sqrt{2}$ is an irrational number. Hence $-\sqrt{2}$ is also an irrational number. Then sum is $\sqrt{2} + (-\sqrt{2})$ which is equal to 0, a rational number. Similarly $\sqrt{2}\,.\,\sqrt{2} = (\sqrt{2})^2 = 2$, which is, again, a rational number.
In the first case the sum of two irrational numbers is a rational number and in the second case, the product of two irrational numbers is rational
On the other hand, the sum of the two irrational numbers $\sqrt{2}$ and $\sqrt{3}$ is irrational. To prove this suppose that $\sqrt{2} + \sqrt{3}$ is not irrational. Then $\sqrt{2} + \sqrt{3}$ must be rational. Since the square of a rational number is rational, $(\sqrt{2} + \sqrt{3})^2$ should be rational
Thus $(\sqrt{2} + \sqrt{3})^2 = 2 + 3 + 2\sqrt{2}.\sqrt{3}$ is a rational number, i.e., $5 + 2\sqrt{6}$ is a rational number. Now, $\sqrt{6}$ is irrational, hence $2\sqrt{6}$ is irrational. Since the sum of

a rational number with an irrational number is irrational [Example 4 (ii)], it follows that $5 + 2\sqrt{6}$ is irrational, which implies that $(\sqrt{2} + \sqrt{3})^2$ is irrational This contradiction proves that $\sqrt{2} + \sqrt{3}$ is an irrational number

Again, the product of the two irrational numbers $\sqrt{2}$ and $\sqrt{3}$ is irrational, since $\sqrt{2}\ \sqrt{3} = \sqrt{6}$ which is irrational, as 6 is not a perfect square

Thus, we see that the sum and product of two irrational numbers may be rational in some cases and irrational in some other cases.

Example 2.6: Give a rational approximation to $\sqrt{3}$ correct to two places of decimals.

Solution: We know that $1^2 = 1 < 3 < 4 = 2^2$

Taking positive square roots we get

$$1 < \sqrt{3} < 2.$$

Next, $(1.7)^2 = 2.89 < 3 < 3.24 = (1.8)^2$

Taking positive square roots, we have

$$1.7 < \sqrt{3} < 1.8.$$

Again, $(1.73)^2 = 2.9929 < 3 < 3.0276 = (1.74)^2$

Taking positive square roots, we obtain

$$1.73 < \sqrt{3} < 1.74.$$

Hence the required approximation is 1.73.

Note: $\sqrt{3} = 1.7320508\ldots$

Example 2.7: Determine, on the number line, the point which represents the irrational number $\sqrt{3}$.

Solution: Mark the points 0, 1, 2 on the number line l.

Let A denote the point 0 and B denote the point 1 Draw a straight line perpendicular to $\overline{AB}$ at the point B and let C be a point on this perpendicular such that BC = 1 unit in length. Join AC Then the length of $\overline{AC}$ in units = $\sqrt{2}$. Again, draw a straight line perpendicular to $\overline{AC}$ at the point C and let D be a point on this perpendicular such that the length of $\overline{CD}$ = 1 unit. Join the points A and D by a straight line. Then AD = $\sqrt{3}$ units. (Why?) Now, with A as centre and AD as radius draw an arc of a circle intersecting the number line l in a point P, as shown in the figure (Fig. 2.6). Then $\overline{AP}$ = $\sqrt{3}$, and the point P represents the irrational number $\sqrt{3}$, because AP = AD = $\sqrt{3}$.

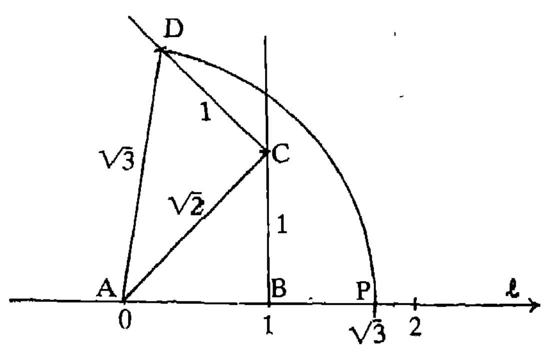


Fig. 2.6

2.11 The Number π

It is well known that the ratio of the length of the circumference of a circle to the length of its diameter is always constant. It is also known that this constant is an irrational number The proof of this fact cannot be given here. This irrational number is denoted by the Greek letter π. Since π is irrational, its decimal representation will be non-terminating and non-repeating. The value of π to a few places of decimals is given below.

$$\pi = 3.14159265\ldots$$

It may be remarked that the rational number π cannot be obtained by the process of root-extraction The rational number $\frac{22}{7}$ is very often taken to be an approximate value of π, but it should be remembered that $\frac{22}{7}$ is not equal to π, and it approximates π to two places of decimal only, since

$$\frac{22}{7} = 2.14285714\ldots$$

The great Indian mathematician, Aryabhata, (born, A.D. 476) gave, in the year A D. 499, the following approximate value of π. He observed that $\pi = \frac{62832}{20000}$ approximately. Changing this fraction into decimals, we get $\pi = 3.1416$ approximately. A closer approximation of π is $\frac{355}{113}$, which equals to 1.14159292 . .

Using an identity of the great mathematical genius, Srinivasa Ramanujan (1887-1920) of India, mathematicians have been able to calculate the value of π correct to millions of places of decimal.

Example 2.8: Find a rational number and also an irrational number between the numbers a and b given below:

$$a = 0.101001000100001\ldots$$
$$b = 0.1001000100001\ldots$$

Solution: The numbers a and b are both irrational numbers, since their decimal representations are non-terminating and non-repeating. Also $a > b$ since in the third place of decimal a has a 1 and b has a zero. Now consider the number c given by

$$c = 0.101$$

c is a rational number as it has a terminating decimal representation. Also $c > b$, since b has a zero in the third place of decimal and c has a 1.

Again, observe that the digits in the first three places of the decimals of a and c are the same. But the decimal for c has zeros in all the places after the third place of decimal whereas the decimal representation of a has a 1 in the sixth place. Hence $c < a$. Thus $b < c < a$.

Let us now find an irrational number between a and b. Consider the number d given by

$$d = 0.1002000100001\ldots$$

Evidently, the number d is irrational. Also $d > b$, since in the first three places of the decimal representations of d and b they have the same digits but in the fourth place d has a 2 whereas b has only a 1. In the remaining places both of them have the same digits. Comparing a and d we see that in the first two places of their decimals both a and d have the same digits, but in the third place of its decimal a has a 1 whereas d has a zero. Hence $a > d$. Thus the irrational number d is such that $b < d < a$.

We can thus say that between any two distinct irrational numbers there are as many rational numbers as we like, and also there are as many irrational numbers as we like. Stated in different words, between any two distinct irrational numbers there are infinitely many distinct rational numbers and infinitely many distinct irrational numbers.

Example 2.9: Find one irrational number between the number a and b given below:

$$a = 0.1111\ldots = 0.\dot{1}$$
$$b = 0.1101$$

Solution: Both the numbers a and b are rational, a having a repeating decimal and b having a terminating decimal. Also

$$b < a.$$

Consider the number c such that

$$c = 0.111101001000100001\ldots$$

The number c is evidently an irrational number, having a non-terminating and non-repeating decimal.

In the first two places of their decimals, c and b have the same digits, but c has a 1 and b has a zero, in the third place.

Hence,

$$b < c$$

Also c and a have the same digits in the first four places of their decimal representations but in the fifth place c has a zero and a has a 1. Also in the sixth place of their decimals a and c have the same digit namely 1, but in each of the seventh and eighth places c has a zero whereas a has a 1.

Thus, $c < a$

Hence, $b < c < a$

In this manner we can show that between any two distinct rational numbers there exists an irrational number. Hence we can say that between any two distinct rational numbers, there can be found as many irrational numbers as we like. In other words, there are infinitely many distinct irrational numbers between any two distinct rational numbers.

Exercises 2.2

1. Identify on the number line the points x, which satisfy the following conditions:
   (i) $|x| \leq 5$
   (ii) $1 < |x| < 2$
   (iii) $|x| = \sqrt{2}$
   (iv) $\frac{|x|}{2} = 3$
2. Find the distance between
   (i) -2 and -4
   (ii) -2 and 4
   (iii) -7 and $|-7|$
3. Replace '*' between the following pairs of numbers by one of the three symbols: ">," "=," and "<".
   (i) $|7-2|$ * $(|7| - |2|)$
   (ii) $|2-7|$ * $(|2| - |7|)$
   (iii) $|8-(-3)|$ * $(|8| - |-3|)$
   (iv) $|-8-3|$ * $|-8| - |3|$
4. Using the formula $(a-b)^2 = a^2 - 2ab + b^2$, find the expansion of $[\,|a| - |b|\,]^2$ and deduce from it that $2\,|a|\,|b| \leq a^2 + b^2$ for any two rational numbers a and b.
   When will the sign of equality occur in the above relation?
5. Find the points x on the number line such that
   (i) $|x-5| < 3$
   (ii) $|x-5| = 3$
   (iii) $|x-5| > 3$
6. Using the method given in Example 2.6, find the values of $\sqrt{5}$ and $\sqrt{7}$ correct to two places of decimal.
7. Prove that $\sqrt{3} - \sqrt{2}$ is irrational
8. Prove that $\sqrt{3} + \sqrt{5}$ is an irrational number.
9. Give two rational numbers lying between 0.232332333233332 . . . and 0.21211211121111 . .
10. Examine, whether the following numbers are rational or irrational:
    (i) $(\sqrt{2} + 2)^2$ (ii) $(2-\sqrt{2})(2 + \sqrt{2})$
    (iii) $(\sqrt{2} + \sqrt{3})^2$ (iv) $\frac{6}{3\sqrt{2}}$
11. Find two irrational numbers between 2 and 2.5.
12. Find two irrational numbers between 0.1 and 0.12.
13. Give two irrational numbers lying between $\sqrt{2}$ and $\sqrt{3}$.

quantity whose value depends on varying value of another.

2.12 Functions

The concept of function is fundamental in mathematics. Before we formally define this concept, it will be helpful to explain the basic ideas by means of an example. Suppose that there are 30 students in a class and that each student brings, as a rule, 5 books with him/her in the class every day. Can you find the correct answer to the question as to how many books will be brought by the students on the first day of the next month? No, you cannot know the correct answer, for the number of books will depend on the number of students present on the first day of the next month, and this number cannot be known in advance. If the number of students present on that day is 25, then the number of books will be 125. If, however, the number of students present is 28, the number of books will be 140, and so on If we denote the number of students present by x, then the number of books will be $5x$ Now, x can take integral values between 0 and 30, i e , $0 \leq x \leq 30$, and $5x$ will take integral values between 0 and 150, i.e. $0 \leq 5x \leq 150$. Let us denote the set of non-negative integers between 0 and 30 by X and the set of non-negative integers between 0 and 150 by Y. Then

$$X = \{x \ \ x \text{ is integer and } 0 \leq x \leq 30\}$$

$$\text{and } Y = \{y \text{ is integer and } 0 \leq y \leq 150\}$$

Now with every element $x \in X$ we associate an element $y \in Y$ such that $y = 5x$. This relationship is described by saying that a function has been defined on the set X and this function takes values in the set Y.

This example leads to the following definition.

Definition

Let X and Y be any two non-empty sets. If with every element $x \in X$ is associated in some manner, one and only one element of Y, then this association determines a function from X to Y

Notice that the important idea is to associate every element of X with one element of Y. This association is usually defined by some rules, as in the previous example. Secondly, whereas to each element of X is associated one element of Y, every element of Y need not get associated to some element of X. In the above example, X is the set of number of students and Y is the set of number of books Now, the number $9 \in Y$, but 9 is not associated with any element of X Only the multiples of 5 will get associated with some elements of X, since $y = 5x$. Thirdly, the same element of Y may get associated to several elements of X, but to each element of X will be associated exactly one element of Y.

Usually the letter 'f' denotes a function, but the letters 'g', 'h', 'k', 's' are also used to denote function We, then write $x \in X$ is associated to $f(x) \in Y$. We also write $f : x \to y$. The set X is called the **Domain** field of action of the function and the set Y is called the **Co-Domain**. The word 'map' is also used and we say that a function maps the set X into the set Y, or it is said that a mapping of X into Y is defined. In symbols we write

$$X \xrightarrow{f} Y,$$

Which is read as "f is function defined on X to Y" or "f maps X into Y."

If a function f associates with an element $x \in X$ an element $y \in Y$, we say "$y = f(x)$" which is read as "y is f of x," or "y is the value of f at x." The element y is called the *image* of x under f.

We will primarily be concerned, in this book, with functions whose domains and co-domains are sets of real numbers. Such functions are called **Real Functions.** The domain and co-domain are normally specified while defining a function.

Example 2.10: If a real function f is defined such that its value at x is given by $f(x) = \sqrt{x}$, find the domain of f. Find $f(4)$, $f(3)$ and $f(8)$

Solution: The value at x is $\sqrt{x}$ and $\sqrt{x}$ is defined only when x is non- negative. Thus the domain is the set of all non-negative real numbers

$$f(4) = \sqrt{4} = 2$$
$$f(3) = \sqrt{3}$$
$$f(8) = \sqrt{8} = 2\sqrt{2}$$

Example 2.11: If a real function g is defined such that its value at x is given by $g(x) = 4x-3$, find the domain of g. Evaluate $g(1)$ and $g(-3)$

Solution: Since $g(x) = 4x - 3$ and $4x - 3$ is defined for all real values of x, we have the domain of g as the set of all real numbers

$$g(1) = (4 \times 1-3) = 1 \,;\, g(-3) = 4 \times (-3)-3$$
$$= -12-3$$
$$= -15$$

Note: If f is a real function of x, where the letter x can assume any value in the domain of f, x is called a variable, or an indeterminate. If we take $y = f(x)$, then the value of y will depend on the value of x. For this reason y is called the *dependent variable* and x is called the *independent variable.*

Example 2.12: If a real function h is defined such that its value at x is given by $h(x) = -3x^2 + 7$ find the domain of h. Evaluate $h(1)$, $h(-1)$ and $h(a)$, for real number a.

Solution: Since $h(x) = -3x^2 + 7$ and $-3x^2 + 7$, which is well defined for every real value x, the domain of h is the set of all real numbers.

$$h(1) = -3 \times 1^2 + 7 = -3 + 7 = 4$$
$$h(-1) = -3 \times (-1)^2 + 7 = -3 + 7 = 4$$
$$h(a) = (-3) \times a^2 + 7 = -3a^2 + 7$$

Example 2.13: If the real function f is defined such that its value at x is $f(x) = \sqrt{(x-3)}\,\sqrt{(5-x)}$, find its domain.

Solution: $f(x) = \sqrt{(x-3)}\,\sqrt{(5-x)}$

We have to consider those values of x, for which $\sqrt{(x-3)}\,\sqrt{(5-x)}$ is defined, i e. those values of x for which

$$x-3 \geq 0 \text{ and } 5-x \geq 0$$

i.e. $x \geq 3$ and $x \leq 5$, therefore, $3 \leq x \leq 5$

Hence the domain of this function is the set of all real numbers between 3 and 5 including 3 and 5. We can write

$$\text{Domain of } f = \{x : x \text{ is real and } 3 \leq x \leq 5\}$$

2.13 Rectangular Coordinate System and Graphs

We get a better understanding of the behaviour and properties of a function from its pictorial representation For a real function usually the graph in a rectangular coordinate system is used for this purpose

We draw two perpendicular lines, one horizontal and the other vertical The point of intersection is denoted by O and is called the origin of the coordinate system. The horizontal line is called the x-axis and the vertical line the y-axis (See figure below).

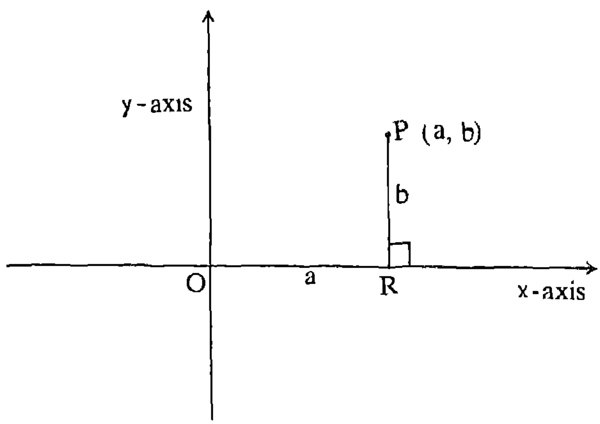


Fig 2 7

We know that every point of a line represents a real number, and conversely, every real number is represented by a unique point on the line. Thus the points on the x-axis will represent real numbers and the points on the y-axis will also represent real numbers We choose a convenient unit for measurement along x-axis and along y-axis (The units chosen for x and y axes should be the same). Let P be any point in the plane. We draw a perpendicular from P on to the x-axis Let it meet the x-axis at R (See figure 2.7). Then the length of the line segment OR (in the unit chosen for x-axis) is called the *x-coordinate* of the point P, or the *abscissa* of P. The length of the line segment PR (in the unit chosen for y-axis) is called the *y-coordinate*, or the *ordinate* of the point If a is the x coordinate of P and b is the y-coordinate of P, then we say that the ordered pair (a, b) are the rectangular coordinates of P (a, b) is called an ordered pair because a and b are placed in a specific order Thus (b, a) are the coordinates of a point different from P. The word "rectangular" is used because the *x-axis* and the y-axis are at right angles If the point R is to the right of O, then the x-coordinate of P is positive; and if R is to the left of O, the x-coordinate of P is negative. If R is at O then the x-coordinate is zero Similarly, if P is above the x-axis, the y-coordinate is positive, if P is below the x-axis, the y-coordinate is negative; and if P is on the x-axis then the y-coordinate is zero.

Note: If R is at O, then P must be on the y-axis, and if P is on the y-axis then its x-coordinate is zero

Example 2.14: In Figure 2.8, write the rectangular coordinates of the points P, S, T, U

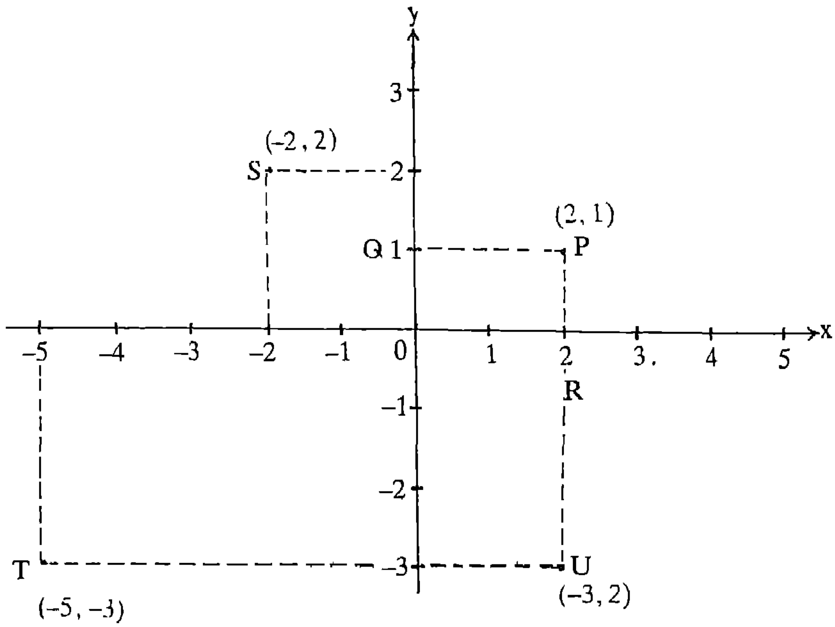


Fig. 2 8

Solution: If we draw perpendicular from P to the x-axis, it meets the x-axis in R, and R is 2 units to the right of O.
Therefore, the x-coordinate of P is 2 The perpendicular from P on the y-axis meet it in the point Q. The line segment OQ has length 1
So, the y-coordinate is 1 Thus, the rectangular coordinates of P are (2, 1). Similarly we show that S has the rectangular coordinates $(-2, 2)$, the rectangular coordinates of T are $(-5, -3)$, and of U are $(2, -3)$. Hence onwards, we will use only the term "coordinates" instead of "rectangular coordinates," as we will be always concerned with rectangular coordinates in this chapter. The part of the x-axis to the right of O is called the "negative x-axis" and the part below the point O is called the "negative y-axis."

Example 2.15: What are the coordinates of O, the origin?
Solution: The coordinates of O are (0, 0). (Why?)

Example 2.16: If a point has coordinates $(0, -4)$, does it lie on the x-axis or the y-axis?
Solution: Since the x-coordinate is zero, it lies on the y-axis.

Example 2.17: Does the point $(2, -3)$ lie above the x-axis or below the x-axis?
Solution: Since the y-coordinate is negative, it lies below the x-axis.

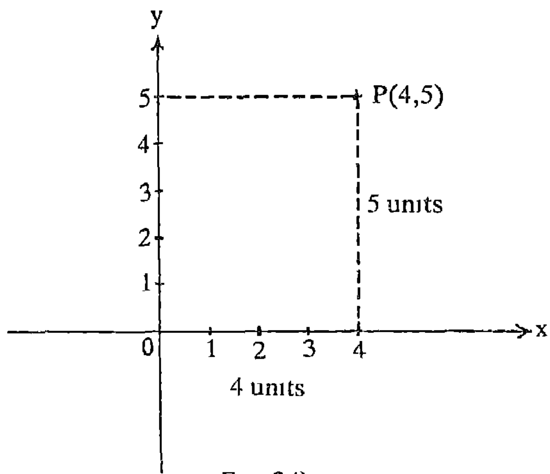


Fig 2.9

We have seen above that every point P in the plane can be given rectangular coordinates. Similarly, given the rectangular coordinates, say (4, 5) we can mark a point having these coordinates. How do we do this? First draw two perpendicular lines, one horizontal line and the other a vertical line. Mark the point of intersection as O. Choose a unit for measuring along the x-axis and for the y-axis. Now since the x coordinate of (4, 5) is 4, which is positive, start from O and measure 4 units to the right of O on the x-axis. From this point draw a line parallel to the y-axis. Again since the y-coordinate is positive, measure 5 units on the y-axis above O and draw a line from this point parallel to the x-axis. (Upwards because the y coordinate is positive). The point of intersection of these lines is the point with coordinates (4, 5). (See Fig 2.9)

Example 2.18: Represent the points with coordinates $(-3, 5)$ and $(3, -5)$ and $(-2, -4)$ in a rectangular coordinate system.
Solution: (See Fig.2.10)

Note: The perpendicular lines in the rectangular coordinate system, namely the x, y axes divide the plane into 4 parts. The upper right hand quarter is called the "*first quadrant*," the upper left hand quarter is called the "*second quadrant*," the bottom left hand quarter is called the "*third quadrant*," and the bottom right hand quarter is called the "*fourth quadrant*."

If a point has x-coordinate positive and y-coordinate positive, it is in first quadrant; if it has x-coordinate negative and y-coordinate positive, it is in second quadrant, if it has both x and y coordinates negative, it is in third quadrant; and if it has x-coordinate positive and y-coordinate negative, it is in fourth quadrant.

Example 2.19: Determine the quadrant in which the points with the following coordinates lie.

(*i*) $(3, -7)$ (*ii*) $(-4, 2)$ (*iii*) $(2, 3)$ (*iv*) $(-1, -\frac{1}{2})$

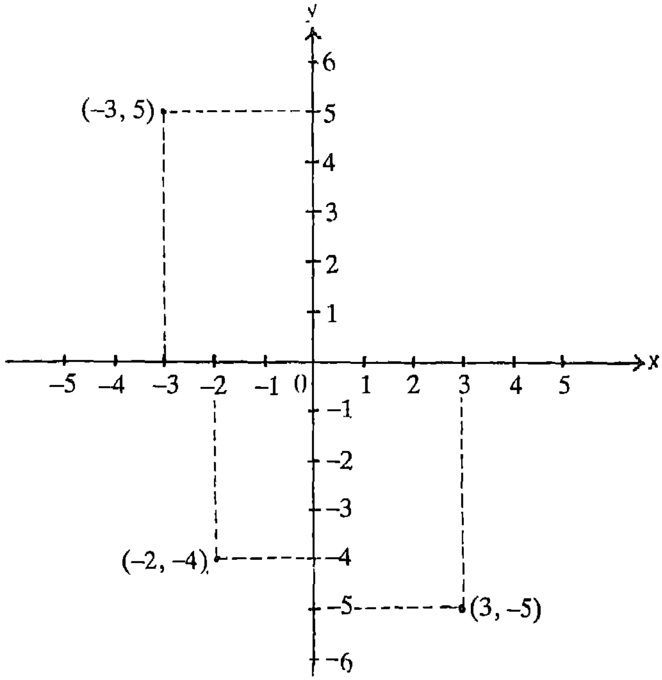


Fig 2.10

Solution: (i) Abscissa = 3, which is positive.
Ordinate = -7, which is negative.
So the point lies in the fourth quadrant

Similarly, (ii) lies in the second quadrant,
(iii) lies in the first quadrant, and
(iv) lies in the third quadrant.

Graphs of Functions

We shall now learn how to sketch the graphs of some real functions. Examine the following examples

Example 2.20: Let f be a function, defined as $f(x) = 3x - 2$. Since the value $3x - 2$ at x is defined for all real values of x, the domain of the function is the set of all real numbers. We now choose various values of x in the domain and calculate the values of $f(x)$ and we tabulate as follows:

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-14	-11	-8	-5	-2	1	4	7	10

We now plot the points, in a rectangular coordinate system, with abscissa as different values of x in the table and as ordinates the corresponding values of $f(x)$. So we plot the points $(-4, -14)$, $(-3, -11)$, $(-2, -8)$, $(-1, -5)$, $(0, -2)$, $(1, 1)$, $(2, 4)$, $(3, 7)$ and $(4, 10)$ and then join by a smooth curve (straight or curved) to pass through these points The line, thus obtained, is the graph of the given function. (See Fig 2 11)

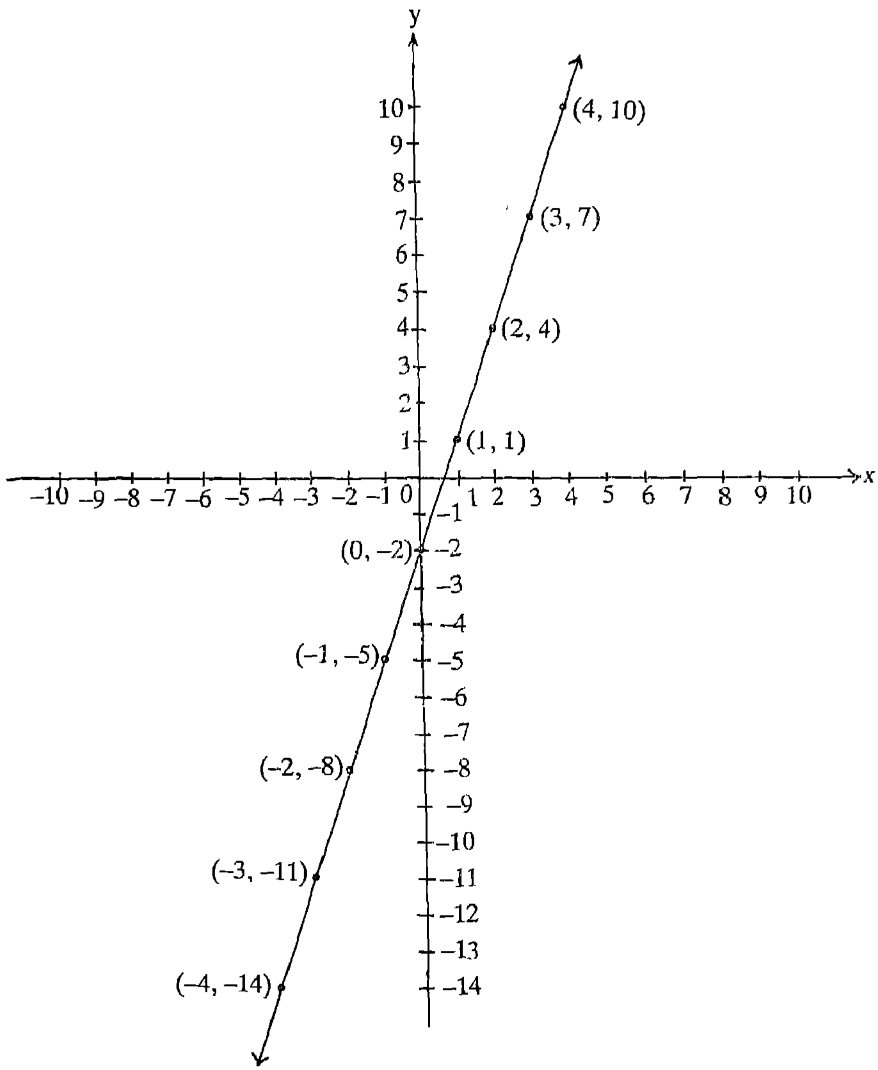


Fig 2 11

Note: The graph of the function $f(x) = mx + c$, where m and c are real numbers, is a straight line. Functions of this form are called "linear functions"

Example 2.21: Let f be a function, defined as $f(x) = x^2$. Sketch the graph of f.

Solution: Since x^2 is defined for all real values of x, the domain of f is the set of all real numbers

Again, we choose various values of x and tabulate the corresponding values of $f(x)$.

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
$f(x)$	25	16	9	4	1	0	1	4	9	16	25

Proceeding as in the previous example, we plot the points

$(-5, 25)$, $(-4, 16)$, $(-3, 9)$, $(-2, 4)$, $(-1, 1)$, $(0, 0)$, $(1, 1)$, $(2, 4)$, $(3, 9)$, $(4, 16)$ and $(5, 25)$.

Joining all these points by a smooth free hand curve we get the graph of the function as given in Fig. 2 12.

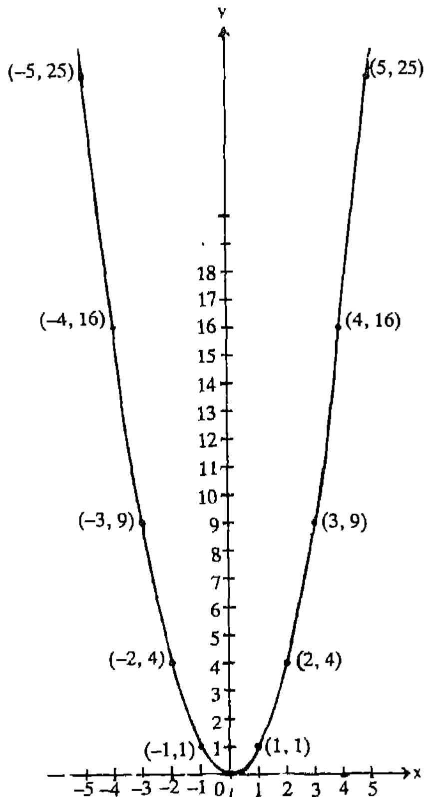


Fig 2 12

in example 2.20 we obtained the graph as a straight line whereas the graph in this example is a curve different from a straight line

Exercises 2.3

1. Find the domain of the following real functions:

   (*i*) $f(x) = 3x + 4$ (*ii*) $f(x) = \sqrt{x-3}$

   (*iii*) $f(x) = \sqrt{x^2 + 2}$ (*iv*) $f(x) = \sqrt{(x-3)(10-x)}$

   (*v*) $f(x) = \sqrt{x} + \sqrt{8-x}$

2. For the functions given in question 1 above evaluate $f(1), f(2), f(-1)$ wherever they are defined.

3. Represent the following points in a rectangular coordinate system.

   (*i*) $(1, 0)$ (*ii*) $(-3, 6)$ (*iii*) $(4, -\frac{3}{2})$

   (*iv*) $(-\frac{1}{2}, -\frac{2}{3})$ (*v*) $(-1, -2)$ (*vi*) $(0, \frac{1}{2})$

4. Which of the following statements are true and which are false?

   (*i*) The point $(-2, 0)$ lies on the y-axis.

   (*ii*) The point $(0, -4)$ lies on the y-axis

   (*iii*) The point $(-1, 2)$ lies below the y-axis

   (*iv*) The point $(2, -3)$ lies below the y-axis.

   (*v*) The point $(-1, -1)$ lies on the x-axis.

   (*vi*) The point $(2, -3)$ lies in the third quadrant.

5. State in which quadrant do the following points lie:

   (*i*) $(1, 1)$ (*ii*) $(-\frac{3}{2}, -\frac{2}{3})$ (*iii*) $(2, -3)$

   (*iv*) $(-5, -\frac{5}{7})$ (*v*) $(-3, 3)$

6. Sketch the graph of the following functions.

   (*i*) $f(x) = 2x + 3$ (*ii*) $f(x) = -\frac{1}{2}x - \frac{3}{2}$

   (*iii*) $f(x) = x^2 - 3$ (*iv*) $f(x) = -x^2 + 2$

CHAPTER 3

Surds

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3.1 Introduction

You may recall that the set R of all real numbers is the union of two disjoint subsets, the set of all rational numbers and the set of all irrational numbers. Thus, a real number which is not a rational number, is an irrational number.

In particular, you know that there is no rational number whose square is 2. However, there is a real number whose square is 2. In decimal form, this real number is written as

$$1.4142135\ldots$$

a non-terminating non-repeating decimal. We agreed to denote this real number in short form by the symbol

$$\sqrt{2} \quad \text{or} \quad 2^{\frac{1}{2}}$$

Similarly, the real irrational number whose cube is 2, will be denoted by the symbol $\sqrt[3]{2}$ or $2^{\frac{1}{3}}$.

Numbers of the type

$$\sqrt{2}, \sqrt[3]{2}, \sqrt{3}, \sqrt[4]{5}, \sqrt[5]{21}, \text{ etc}$$

are called *surds* or *radicals*. Each of them represents an irrational number. In general, if a is a positive rational number, which cannot be expressed as the nth power of some rational number, then the irrational number $\sqrt[n]{a}$ or $a^{\frac{1}{n}}$, that is, the positive nth root of a, is called a *surd* or *a radical*. The symbol $\sqrt[n]{\ }$ is called the radical sign, n is called the *order* of the surd (or radical) and a is called the *radicand*.

It may be noted that by the statement '$\sqrt[n]{a}$ is a surd,' it is understood that

(i) a is a rational number and

(ii) $\sqrt[n]{a}$ is an irrational number.

We consider an expression which is expressible in the form $\sqrt[n]{a}$ where a is rational and $\sqrt[n]{a}$ is irrational, to be a surd. For example, $\sqrt{\sqrt{2}}$ appears to be not a surd because it is the square root of an irrational number ($\sqrt{2}$) but it is considered as a surd because it can be expressed as $\sqrt[4]{2}$.

Note: For a positive real number a (in particular for a positive rational number) it can be shown that there is a unique real number x which is the positive nth root of a or which is positive and satisfies $x^n = a$. The proof of this result is beyond the scope of the present book.

Examples:

(i) $\sqrt[3]{5}$ is a surd of order 3

(ii) $\sqrt{50}$ is a surd of order 2

(iii) $\sqrt{64}$ is not a surd, since $64 = 8^2$ and therefore $\sqrt{64} = 8$, is a rational number

(iv) $\sqrt{3 + 2\sqrt{2}}$ is an irrational number but it is not a surd, since the number $3 + 2\sqrt{2}$ is not a rational number

Remarks: When the radicand is denoted by a symbol, such as a in $\sqrt[n]{a}$ it is understood that a satisfies the necessary conditions for $\sqrt[n]{a}$ to be called a surd

3.2 Laws of Radicals

(1) You will recall that for any positive integer n and a positive rational number a, the radical $\sqrt[n]{a}$ is the positive nth root of a,

$$(\sqrt[n]{a})^n = a$$

(2) Suppose $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are two radicals of the same order, then

$$(\sqrt[n]{a}\sqrt[n]{b})^n = (\sqrt[n]{a})^n (\sqrt[n]{b})^n$$
$$= ab$$
$$\therefore \sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$$

(3) If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are two radicals of the same order, then

$$\left(\frac{\sqrt[n]{a}}{\sqrt[n]{b}}\right)^n = \frac{(\sqrt[n]{a})^n}{(\sqrt[n]{b})^n} = \frac{a}{b}$$
$$\therefore \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

(4) If m, n are positive integers, then for a positive rational number a,

$$[\sqrt[m]{(\sqrt[n]{a})}]^{mn} = (\sqrt[n]{a})^n = a$$
$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$$
$$\text{Similarly, } \sqrt[n]{\sqrt[m]{a}} = \sqrt[mn]{a}$$
$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a} = \sqrt[n]{\sqrt[m]{a}}$$

(5) If m, n and are positive integers, then for a positive rational number a,

$$\sqrt[n]{a^p} = \sqrt[n]{\sqrt[m]{(a^p)^m}} = \sqrt[mn]{a^{pm}}$$

Observe that the index of the radical and the exponent of the radicand are both multiplied by the same number m.

Examples:

1 $\quad (\sqrt[3]{5})^3 = 5$

2 $\quad \sqrt{125} = \sqrt{25 \times 5} = \sqrt{25}\sqrt{5} = 5\sqrt{5}$

3 $\quad \sqrt[3]{\frac{2}{3}} = \frac{\sqrt[3]{2}}{\sqrt[3]{3}}$

4 $\quad \sqrt[6]{64} = \sqrt[2\times3]{64} = \sqrt{\sqrt[3]{64}} = \sqrt{4} = 2$

5 $\quad \sqrt[5]{\sqrt[3]{3}} = \sqrt[5\times3]{3} = \sqrt[15]{3}$

The laws of radicals are used to simplify a given radical or to reduce two given radicals to the same form For example, we have shown above that

$$\sqrt{125} = 5\sqrt{5}$$

The surd on the right is in its simplest form Similarly,

$$\sqrt[3]{144} = \sqrt[3]{2^4 \times 3^2} = \sqrt[3]{2^3 \times 2 \times 3^2} = 2\sqrt[3]{18}$$

A surd in its simplest form has

(i) no factor which is nth power of a rational number under the radical sign whose index is n,

(ii) no fraction under the radical sign

and (iii) the smallest possible index of this radical

Example 3.1: Express $\sqrt[4]{1875}$ in its simplest form

Solution:
$$\sqrt[4]{1875} = \sqrt[4]{5 \times 5 \times 5 \times 5 \times 3} = \sqrt[4]{5^4 \times 3} = 5\sqrt[4]{3}$$

Example 3.2: Simplify $\sqrt{\frac{125}{63}}$

Solution:
$$\sqrt{\frac{125}{63}} = \frac{\sqrt{125}}{\sqrt{63}} = \frac{\sqrt{5 \times 5 \times 5}}{\sqrt{3 \times 3 \times 7}} = \frac{5\sqrt{5}}{3\sqrt{7}}$$
$$= \frac{5\sqrt{5}}{3\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}} = \frac{5\sqrt{35}}{21}$$

A surd which has a rational factor other than unity, the other factor being irrational, is called a *mixed surd* A surd which has unity as its rational factor, the other factor being irrational, is called a *pure surd*

$\sqrt{10}$, $\quad \sqrt{8}$, $\quad \sqrt[3]{7}$ are pure surds

$2\sqrt{3}$, $\quad 5\sqrt[3]{16}$, $\quad 2\sqrt[3]{7}$ are mixed surds

The laws of radicals enable us to express a pure surd as a mixed surd or vice versa.

Example 3.3: Express $\frac{3}{4}\sqrt{32}$ as a pure surd.

Solution: $\frac{3}{4}\sqrt{32} = \sqrt{\left(\frac{3}{4}\right)^2}\,\sqrt{32} = \sqrt{\frac{9}{16}\times 32}$

$= \sqrt{18}$

Example 3.4: Express $\sqrt[3]{256}$ as a mixed surd in its simplest form.

Solution: $\sqrt[3]{256} = \sqrt[3]{16^2} = \sqrt[3]{2.^3\ 2.^3\ 2.^2}$

$= \sqrt[3]{2^3}\ \sqrt[3]{2^3}\ \sqrt[3]{2^2}$

$= 2\times 2\times \sqrt[3]{4}$

$= 4\sqrt[3]{4}$

Exercises 3.1

1 Express as a pure surd·

(i) $5\sqrt{6}$ (ii) $2\sqrt[3]{4}$ (iii) $3\sqrt[4]{5}$

(iv) $10\sqrt{3}$ (v) $\frac{2}{3}\sqrt{32}$ (vi) $\frac{3}{4}\sqrt{8}$

2. Express as a mixed surd in its simplest form·

(i) $\sqrt{80}$ (ii) $\sqrt[3]{72}$ (iii) $\sqrt[5]{288}$

(iv) $\sqrt[3]{150}$ (v) $\sqrt[5]{320}$ (vi) $5\sqrt[3]{135}$

3.3 Comparison of Surds

Two surds of the same order can be easily compared We just compare their radicands, Thus,

$\sqrt[3]{25} > \sqrt[3]{24}$, $\sqrt[5]{84} > \sqrt[5]{80}$, etc.

If the surds are not of the same order, we first reduce them to the same order.

Example 3.5: Which is greater $\sqrt[3]{3}$ or $\sqrt[4]{5}$?

Solution: L.C.M of 3 and 4 is 12.

Thus, $\sqrt[3]{3} = \sqrt[12]{3^4} = \sqrt[12]{81}$

and $\sqrt[4]{5} = \sqrt[12]{5^3} = \sqrt[12]{125}$

Now, clearly $\sqrt[12]{125} > \sqrt[12]{81}$

$\therefore \sqrt[4]{5} > \sqrt[3]{3}$

Example 3.6: Arrange in descending order of magnitude·

$\sqrt[4]{3},\ \sqrt[6]{10},\ \sqrt[12]{25}$

Solution: L C M. of 4,6 and 12 is 12.

$$\begin{array}{llllll} \text{Hence,} & \sqrt[4]{3} & = & \sqrt[12]{3^3} & = & \sqrt[12]{27} \\ & \sqrt[6]{10} & = & \sqrt[12]{10^2} & = & \sqrt[12]{100} \\ \text{and} & \sqrt[12]{25} & = & \sqrt[12]{25} & & \end{array}$$

Now, arranging in descending order

$$\sqrt[12]{100} > \sqrt[12]{27} > \sqrt[12]{25}$$

$$\therefore \quad \sqrt[6]{10} > \sqrt[4]{3} > \sqrt[12]{25}$$

Exercises 3.2

1 Which is greater?

(*i*) $\sqrt{2}$ or $\sqrt[3]{3}$

(*ii*) $\sqrt{3}$ or $\sqrt[4]{10}$

(*iii*) $\sqrt[4]{5}$ or $\sqrt[3]{4}$

(*iv*) $\sqrt[3]{6}$ or $\sqrt[4]{8}$

(*v*) $\sqrt[8]{12}$ or $\sqrt[4]{6}$

(*vi*) $\sqrt[3]{3}$ or $\sqrt[4]{4}$

2 Arrange in descending order of magnitude

(*i*) $\sqrt[3]{4}, \sqrt[4]{5}, \sqrt{3}$ (*ii*) $\sqrt[3]{2}, \sqrt[4]{3}, \sqrt[6]{4}$

(*iii*) $\sqrt[4]{10}, \sqrt[3]{6}, \sqrt{3}$ (*iv*) $\sqrt[3]{2}, \sqrt[6]{3}, \sqrt[7]{4}$

3.4 Addition and Substraction of Surds

Since surds are real numbers, the distributive law holds.

Thus, (*i*) $5\sqrt{2} + 4\sqrt{2} = (5 + 4)\sqrt{2} = 9\sqrt{2}$

and (*ii*) $4\sqrt{7} + 5\sqrt{7} - 3\sqrt{7} = (4 + 5 - 3)\sqrt{7} = 6\sqrt{7}$

Observe that $5\sqrt{2}$ and $4\sqrt{2}$ have the same irrational factor. Surds having the same irrational factor are called similar surds. Using the distributive law similar surds can be added and subtracted

Example 3.7: Simplify

$$5\sqrt{3} + 2\sqrt{27} + \frac{1}{\sqrt{3}}$$

Solution: Reducing each term to its simplest form

$$5\sqrt{3}+2\sqrt{27}+\frac{1}{\sqrt{3}} = 5\sqrt{3}+2\sqrt{9\times 3}+\frac{\sqrt{3}}{\sqrt{3}\times\sqrt{3}}$$

$$= 5\sqrt{3}+6\sqrt{3}+\frac{1\sqrt{3}}{3}$$

$$= (5+6+\frac{1}{3}\ \sqrt{3}$$

$$= \frac{34\sqrt{3}}{3}$$

Example 3.8: Simplify

$$\sqrt{252}-5\sqrt{6}+\sqrt{294}-3\sqrt{\frac{1}{6}}$$

Solution: Reducing to simplest form

$$\sqrt{252} = \sqrt{4\times 63} = \sqrt{4\times 9\times 7} = 6\sqrt{7}$$
$$\sqrt{294} = \sqrt{2\times 147} = \sqrt{2\times 3\times 49} = 7\sqrt{6}$$
$$\sqrt{\frac{1}{6}} = \sqrt{\frac{6}{36}} = \frac{\sqrt{6}}{6}$$

$\therefore$ The given numerical expression

$$= \sqrt{252}-5\sqrt{6}+\sqrt{294}-3\sqrt{\frac{1}{6}}$$
$$= 6\sqrt{7}-5\sqrt{6}+7\sqrt{6}-\frac{3}{6}\ \sqrt{6}$$
$$= 6\sqrt{7}+(7-5-\frac{1}{2})\sqrt{6}$$
$$= 6\sqrt{7}+\frac{2}{3}\sqrt{6}$$

Exercises 3.3

Simplify by combining similar terms.

(1) $5\sqrt{2}+20\sqrt{2}$

(2) $2\sqrt{3}+\sqrt{27}$

(3) $4\sqrt{3}-3\sqrt{12}+2\sqrt{75}$

(4) $\sqrt{8}+\sqrt{32}-\sqrt{2}$

(5) $\sqrt{45}-3\sqrt{20}+4\sqrt{5}$

(6) $4\sqrt{12}-\sqrt{50}-7\sqrt{48}$

(7) $2\sqrt[3]{4}+7\sqrt[3]{32}-\sqrt[3]{500}$

(8) $2\sqrt[3]{40}+3\sqrt[3]{625}-4\sqrt[3]{320}$

(9) $3\sqrt{147}-\frac{7}{3}\sqrt{\frac{1}{3}}+7\sqrt{\frac{1}{3}}$

(10) $\sqrt[4]{81}-8\sqrt[3]{216}+15.\sqrt[5]{32}+\sqrt{225}$

3.5 Multiplication and Division of Two Surds

Surds of the same order can be multiplied according to the following law

$$\sqrt[n]{a}. \times \sqrt[n]{b} = \sqrt[n]{ab}$$

Thus, we have

(*i*) $\sqrt[3]{18} \times \sqrt[3]{15} = \sqrt[3]{18 \times 15} = \sqrt[3]{270}$

(*ii*) $\sqrt[5]{24} \div \sqrt[5]{6} = \sqrt[5]{\frac{24}{6}} = \sqrt[5]{4}$

If the surds to be multiplied are not of the same order, it is necessary to bring them to the same order, before applying the above rule

Example 3.9: Multiply $\sqrt[3]{7}$ by $\sqrt{2}$

Solution: L C M of 2 and 3 is 6

Hence, $\sqrt[3]{7} = \sqrt[6]{7^2} = \sqrt[6]{49}$

and $\sqrt{2} = \sqrt[6]{2^3} = \sqrt[6]{8}$

$\therefore \sqrt[3]{7} \times \sqrt{2} = \sqrt[6]{49} \times \sqrt[6]{8} = \sqrt[6]{392}$

Example 3.10: Divide $\sqrt{24}$ by $\sqrt[3]{200}$

Solution: L.C.M of 2 and 3 is 6

$$\sqrt{24} = \sqrt[6]{(24)^3} = \sqrt[6]{13824}$$

$$\text{and } \sqrt[3]{200} = \sqrt[6]{(200)^2} = \sqrt[6]{40000}$$

$$\therefore \sqrt{24} \div \sqrt[3]{200} = \sqrt[6]{13824} \div \sqrt[6]{40000}$$

$$= \sqrt[6]{\frac{13824}{40000}}$$

$$= \sqrt[6]{\frac{216}{625}}$$

Exercises 3.4

Simplify and express the result in its simplest form

(1) $\sqrt{14} \times \sqrt{21}$

(2) $\sqrt{15} \times \sqrt{7}$

(3) $\sqrt[3]{4} \times \sqrt[3]{22}$

(4) $4\sqrt{12} \times 7\sqrt{6}$

(5) $\sqrt[3]{2} \times \sqrt{5}$

(6) $\sqrt[3]{2} \times \sqrt[4]{3}$

(7) $4\sqrt{28} \div 3\sqrt{7}$

(8) $\sqrt[6]{12} \div (\sqrt{3}\sqrt[3]{2})$

(9) $\sqrt{2}\ \sqrt[3]{3}.\ \sqrt[4]{4}$

(10) $\sqrt[3]{2}.\ \sqrt[4]{2}.\ \sqrt[12]{32}$

3.6 Rationalisation of Surds

When the product of two surds is a rational number, then each of the two surds is called a *Rationalising factor* (R.F.) of the other

(i) Since, $2\sqrt{5}\times\sqrt{5}=10$
$\therefore \sqrt{5}$ is a R.F. of $2\sqrt{5}$

(ii) As $(\sqrt{3}+\sqrt{2})\ (\sqrt{3}-\sqrt{2})=(\sqrt{3})^2-(\sqrt{2})^2=3-2=1$
$\therefore \sqrt{3}-\sqrt{2}$ is a R.F. of $\sqrt{3}+\sqrt{2}$

(iii) $(\sqrt{3}+\sqrt{10}-\sqrt{5})\ (\sqrt{3}+\sqrt{10}+\sqrt{5})=(\sqrt{3}+\sqrt{10})^2-(\sqrt{5})^2$
$=13+2\sqrt{30}-5=8+2\sqrt{30}$
Also, $(8+2\sqrt{30})\ (8-2\sqrt{30})=64-120=-56$
Hence, $(\sqrt{3}+\sqrt{10}-\sqrt{5})\ (\sqrt{3}+\sqrt{10}+\sqrt{5})\ (8-2\sqrt{30})=56.$
$\therefore (\sqrt{3}+\sqrt{10}+\sqrt{5})\ (8-2\sqrt{30})$ is a R.F. of $\sqrt{3}+\sqrt{10}-\sqrt{5}.$

Remarks: The rationalising factor of a given surd is not unique. If one R F. of a surd is known, then the product of this factor by a rational number is also a R F. of the given surd. It is convenient to use the simplest of all Rationalising factors of a given surd.

3.6 (i) Rationalisation of Monomial Surds

Example 3.11: Find the simplest rationalising factor of $\sqrt{32}$.

Solution: $\sqrt{32} = \sqrt{16\times 2} = 4\sqrt{2}$
Now $4\sqrt{2}\times\sqrt{2}=4\times\sqrt{2\times 2}=8$
$\therefore \sqrt{2}$ is the simplest rationalising factor of $\sqrt{32}$.

Example 3.12: Find the simplest rationalising factor of $\sqrt[3]{72}$.

Solution: $\sqrt[3]{72} = \sqrt[3]{2\times 2\times 2\times 3\times 3} = 2^{\frac{3}{3}}\ 3^{\frac{2}{3}} = 2.3^{\frac{2}{3}}$
Now $(2.3^{\frac{2}{3}})\ (3^{1-\frac{2}{3}}) = 2.3 = 6$
$\therefore 3^{\frac{1}{3}}$ or $\sqrt[3]{3}$ is the simplest rationalising factor of $\sqrt[3]{72}$.

Example 3.13: Find a rationalising factor of $\sqrt[5]{a^2\ b^3\ c^4}$
where *a,b,c,* are rational numbers.

Solution: $\sqrt[5]{a^2 b^3 c^4} = a^{\frac{2}{5}}\ b^{\frac{3}{5}}\ c^{\frac{4}{5}}$
Now $(a^{\frac{2}{5}}\ b^{\frac{3}{5}}\ c^{\frac{4}{5}})\ (a^{\frac{3}{5}}\ b^{\frac{2}{5}}\ c^{\frac{1}{5}}) = a^1\ b^1\ c^1 = abc$
$\therefore a^{\frac{3}{5}}\ b^{\frac{2}{5}}\ c^{\frac{1}{5}}$ is a R.F. of $\sqrt[5]{a^2\ b^3\ c^4}$.
i.e., $\sqrt[5]{a^3\ b^2\ c}$ is a R.F. of $\sqrt[5]{a^2\ b^3\ 1c^4}$

Example 3.14: Rationalise the denominator in $\frac{2}{\sqrt{7}}$.

Solution: The simplest R F of $\sqrt{7}$ is $\sqrt{7}$ itself

$$\therefore \frac{2}{\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}} = \frac{2\sqrt{7}}{7} = \frac{2}{7}\sqrt{7}$$

Example 3.15: Given that $\sqrt{5} = 2\ 236$ approx., find to three places of decimals, the value of $\frac{2.}{\sqrt{5.}}$

Solution:

$$\frac{2}{\sqrt{5}} = \frac{2}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{5} = \frac{2}{5}\sqrt{5}$$

Since $\sqrt{5} = 2\ 236$

$\therefore \quad \frac{2}{\sqrt{5}} = \frac{2 \times 2\ 236}{5} = 8944$

or $\quad \frac{2}{\sqrt{5}} = 0\ 894$ (approx)

Exercises 3.5

1. Write the simplest rationalising factor of

(i) $2\sqrt{2}$ (ii) $\sqrt{10}$

(iii) $\sqrt{75}$ (iv) $2\sqrt[3]{5}$

(v) $\sqrt[3]{36}$ (vi) $\sqrt[3]{32}$

2 Express with a rational demonimator the following surds

(i) $\frac{2}{\sqrt{5}}$ (ii) $\frac{2}{3\sqrt{3}}$

(iii) $\frac{1}{\sqrt{12}}$ (iv) $\frac{\sqrt{2}}{\sqrt{5}}$

(v) $\frac{2\sqrt{7}}{\sqrt{11}}$ (vi) $\frac{3\sqrt[3]{5}}{\sqrt[3]{9}}$

3 Find the value to three places of decimals, of each of the following. It is given that $\sqrt{2} = 1.414$, $\sqrt{3} = 1.732$, $\sqrt{10} = 3.162$ and $\sqrt{5} = 2.236$ (approx.).

(i) $\dfrac{1}{\sqrt{2}}$ (ii) $\dfrac{1}{\sqrt{3}}$

(iii) $\dfrac{1}{\sqrt{10}}$ (iv) $\dfrac{\sqrt{2}+1}{\sqrt{5}}$

(v) $\dfrac{2-\sqrt{3}}{\sqrt{3}}$ (vi) $\dfrac{\sqrt{10}-\sqrt{5}}{\sqrt{2}}$

3.6 (ii) Rationalisation of Binomial Quadratic Surds

Surds of the second order are also called *quadratic surds.* Two binomial expressions containing surds which differ only in the sign (+ or −) connecting them, are said to be conjugate to each other Thus,

$(\sqrt{5}+\sqrt{3})$ and $(\sqrt{5}-\sqrt{3})$ are congugate to each other Similarly, the conjugate of $3+\sqrt{6}$ is $3-\sqrt{6}$ and vice versa

The interesting thing about them is that their product is a rational number For example,

$$(\sqrt{5}+\sqrt{3})\ (\sqrt{5}-\sqrt{3}) = (\sqrt{5})^2-(\sqrt{3})^2 = 2$$

Therefore, the simplest rationalising factor of binomial quadratic surd is its conjugate surd. Thus from the above illustration

$(\sqrt{5}-\sqrt{3})$ is a R.F. of $(\sqrt{5}+\sqrt{3})$.
and $(\sqrt{5}+\sqrt{3})$ is a R.F of $(\sqrt{5}-\sqrt{3})$

We shall illustrate their use by few examples.

Example 3.16: Express $\dfrac{6}{3\sqrt{2}-2\sqrt{3}}$ with a rational denominator

Solution:
$$\frac{6}{3\sqrt{2}-2\sqrt{3}} = \frac{6}{3\sqrt{2}-2\sqrt{3}} \times \frac{3\sqrt{2}+2\sqrt{3}}{3\sqrt{2}+2\sqrt{3}}$$
$$= \frac{6(3\sqrt{2}+2\sqrt{3})}{(3\sqrt{2})^2-(2\sqrt{3})^2} = \frac{6(3\sqrt{2}+2\sqrt{3})}{18-12}$$
$$= 3\sqrt{2}+2\sqrt{3}$$

Example 3.17: Given $\sqrt{2} = 1.4142$ and $\sqrt{3} = 1.7321$, find correct to three place of decimals the value of

$$\frac{4}{3\sqrt{3}-2\sqrt{2}} + \frac{3}{3\sqrt{3}+2\sqrt{2}}$$

Solution: $(3\sqrt{3}+2\sqrt{2})$ and $(3\sqrt{3}-2\sqrt{2})$ are conjugate surds

$$\therefore \quad (3\sqrt{3}+2\sqrt{2})\quad(3\sqrt{3}-2\sqrt{2})$$

$$= \quad (3\sqrt{3})^2-(2\sqrt{2})^2 = 27-8 = 19$$

$$\text{Now} \quad \frac{4}{3\sqrt{3}-2\sqrt{2}} + \frac{3}{3\sqrt{3}+2\sqrt{2}}$$

$$= \quad \frac{4\,(3\sqrt{3}+2\sqrt{2})}{19} + \frac{3\,(3\sqrt{3}-2\sqrt{2})}{19}$$

$$= \quad \frac{1}{19}\quad(12\sqrt{3}+8\sqrt{2}+9\sqrt{3}-6\sqrt{2})$$

$$= \quad \frac{21}{19}\sqrt{3}+\frac{2}{19}\sqrt{2}$$

Substituting the approx values of $\sqrt{3}$ and $\sqrt{2}$ in the above we see that

$$\text{Given surd} = \frac{21}{19}\times 1\,7321 + \frac{2}{19}\times 1.4142$$

$$= \frac{1}{19}\{36.3741+2.8284\}$$

$$= \frac{1}{19}(39.2025)$$

$$= 2\,0631$$

$$= 2.063 \text{ (correct to 3 decimal places)}$$

Example 3.18: Express $\dfrac{3}{\sqrt{3}-\sqrt{2}+\sqrt{5}}$ with a rational denominator

Solution: The denominator is a trinomial surd. We proceed as with a binomial surd, by grouping two of the terms

Thus, $\{\sqrt{3}-\sqrt{2})+\sqrt{5}\}\quad\{(\sqrt{3}-\sqrt{2})\;-\sqrt{5}\}$

$$= (\sqrt{3}-\sqrt{2})^2 - (\sqrt{5})^2 = 5-2\sqrt{6}-5 = -2\sqrt{6}$$

$$\therefore \quad \frac{3}{\sqrt{3}-\sqrt{2}+\sqrt{5}} = \frac{3\,(\sqrt{3}-\sqrt{2}-\sqrt{5})}{\{(\sqrt{3}-\sqrt{2})\;+\sqrt{5})\}\quad\{(\sqrt{3}-\sqrt{2})-\sqrt{5})\}}$$

$$= \frac{3\sqrt{3}-3\sqrt{2}-3\sqrt{5}}{-2\sqrt{6}}$$

$$= \frac{(3\sqrt{3}-\sqrt{3}\sqrt{2}\;-\;3\sqrt{5})\sqrt{6}}{(-2)\times 6}$$

$$= -\frac{1}{12}\{3\sqrt{3}\sqrt{6} - 3\sqrt{2}\sqrt{6} - 3\sqrt{5}\sqrt{6}\}$$

$$= -\frac{1}{12}\{9\sqrt{2} - 6\sqrt{3} - 3\sqrt{30}\}$$

$$= \frac{3\sqrt{30} + 6\sqrt{3} - 9\sqrt{2}}{12}$$

$$= \frac{\sqrt{30} + 2\sqrt{3} - 3\sqrt{2}}{4}$$

Exercises 3.6

1 If both of a and b are rational numbers, find the values of a and b in each of the following equalities:

(*i*) $\frac{\sqrt{3}-1}{\sqrt{3}+1} = a + b\sqrt{3}$ (*ii*) $\frac{3+\sqrt{2}}{3-\sqrt{2}} = a + b\sqrt{2}$

(*iii*) $\frac{5+2\sqrt{3}}{7+4\sqrt{3}} = a + b\sqrt{3}$ (*iv*) $\frac{3-\sqrt{5}}{3+2\sqrt{5}} = a\sqrt{5} - b$

(*v*) $\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}} = a + b\sqrt{15}$ (*vi*) $\frac{\sqrt{2}+\sqrt{3}}{3\sqrt{2}-2\sqrt{3}} = a - b\sqrt{6}$

2 Simplify each of the following by rationalising the denominator:

(*i*) $\frac{5+\sqrt{6}}{5-\sqrt{6}}$ (*ii*) $\frac{\sqrt{7}-\sqrt{5}}{\sqrt{7}+\sqrt{5}}$

(*iii*) $\frac{7+3\sqrt{5}}{7-3\sqrt{5}}$ (*iv*) $\frac{2\sqrt{3}-\sqrt{5}}{2\sqrt{2}+3\sqrt{3}}$

(*v*) $\frac{2\sqrt{6}-\sqrt{5}}{3\sqrt{5}-2\sqrt{6}}$ (*vi*) $\frac{7\sqrt{3}-5\sqrt{2}}{\sqrt{48}+\sqrt{18}}$

3. Rationalise the denominator of

(*i*) $\frac{1}{3+\sqrt{5}-2\sqrt{2}}$ (*ii*) $\frac{1}{\sqrt{3}-\sqrt{2}-1}$

(*iii*) $\frac{1}{\sqrt{6}+\sqrt{5}-\sqrt{11}}$ (*iv*) $\frac{1}{\sqrt{7}+\sqrt{6}-\sqrt{13}}$

4. Simplify each of the following:

(i) $\dfrac{\sqrt{6}}{\sqrt{2}+\sqrt{3}} + \dfrac{3\sqrt{2}}{\sqrt{6}+\sqrt{3}} - \dfrac{4\sqrt{3}}{\sqrt{6}+\sqrt{2}}$

(ii) $\dfrac{7\sqrt{3}}{\sqrt{10}+\sqrt{3}} - \dfrac{2\sqrt{5}}{\sqrt{6}+\sqrt{5}} - \dfrac{3\sqrt{2}}{\sqrt{15}+3\sqrt{2}}$

5 Taking $\sqrt{2}=1.414, \sqrt{3}=1.732, \sqrt{5}=2.236$ and $\sqrt{6}=2.449$ (approx). find the value to three places of decimals of each of the following

(i) $\dfrac{1+\sqrt{2}}{\sqrt{5}+\sqrt{3}} + \dfrac{1-\sqrt{2}}{\sqrt{5}-\sqrt{3}}$

(ii) $\dfrac{2+\sqrt{3}}{2-\sqrt{3}} + \dfrac{2-\sqrt{3}}{2+\sqrt{3}} + \dfrac{\sqrt{3}-1}{\sqrt{3}+1}$

Review Exercises 3.7

1 (a) Which is greater $\sqrt{3}$ or $\sqrt[3]{5}$?

(b) Arrange in ascending order of magnitude:

$\sqrt[4]{10}, \quad \sqrt[3]{6}, \quad \sqrt{3}$

2. Simplify.

(a) $\sqrt[3]{2} + \sqrt[3]{16} - \sqrt[3]{54}$

(b) $\sqrt[3]{32} \times \sqrt[3]{250}$

3 Given that $\sqrt{3}=1.7321$, find correct to 3 places of decimals, the value of

$\sqrt{192} - \frac{1}{2}\sqrt{48} - \sqrt{75}$

4. Simplify by rationalising the denominator·

(i) $\dfrac{4}{\sqrt[3]{16}}$ (ii) $\dfrac{2.\sqrt[3]{3}}{4\sqrt[3]{5}}$

5. Rationalise the denominator of

$$\frac{1}{\sqrt{6}+\sqrt{5}-\sqrt{11}}$$

6. Simplify:

$$\frac{3\sqrt{2}}{\sqrt{6}-\sqrt{3}} - \frac{4\sqrt{3}}{\sqrt{6}-\sqrt{2}} + \frac{2\sqrt{3}}{\sqrt{6}+\sqrt{2}}$$

7. Given $\sqrt{2} = 1.4142, \sqrt{6} = 2.4495$, find correct to three places of decimals, the value of

$$\frac{1}{\sqrt{3}-\sqrt{2}-1}$$

8. Simplify:

$$\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}} + \frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}$$

9. Find rational numbers a and b, such that

$$\frac{4+3\sqrt{5}}{4-3\sqrt{5}} = a + b\sqrt{5}$$

10. If $\sqrt{5} = 2.236$ (approx.), evaluate

$$\frac{3-\sqrt{5}}{3+2\sqrt{5}}$$

correct to three places of decimals.

CHAPTER 4

Polynomials

4.1 Polynomials

We became familiar with the concept of 'function' in Chapter 2. In this chapter we shall discuss a special type of function, known as polynomial. Polynomials can also be considered as special types of algebraic expressions involving only one variable

Consider $p(x) = 3x^2 - 4x + 2$. This is a function of x. If we substitute for x any real value 'a', we get a real number $p(a)$. For example

$$p(2) = 12 - 8 + 2 = 4 + 2 = 6$$
$$p(-1) = 3 + 4 + 2 = 7 + 2 = 9$$

and so on. So $P(x)$ is a function whose domain is the set of all real numbers. Observe that here only non-negative powers of x are involved in each term and the coefficient of each power of x is a real number. Any such function is called a "polynomial" over the real numbers. The word "polynomial" means an algebraic expression consisting of many terms involving powers of the variable. Poly means many or much.

Definition

A function $p(x)$ of the form

$$p(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_nx^n$$

where $a_0, a_1, a_2, \ldots\ldots, a_n$ are real numbers and n is a non-negative integer, is called a "polynomial in x over reals"

The real numbers $a_0, a_1, \ldots\ldots, a_n$ are called the coefficients of the polynomial. If $a_0, a_1, a_2, \ldots\ldots, a_n$ are all integers, we call it a *polynomial over integers*. If they are rational numbers, we can call it a *polynomial over rationals*. For example

$5x^2 - 6x + 3$ is a polynomial over integers,

$\frac{3}{4}x^3 - \frac{4}{3}x^2 + 2x - 1$ is a polynomial over rationals and

$3x^2 + \sqrt{2}x - \sqrt{3}$ is a polynomial over reals

If the variable involved is y then we call it *a polynomial in y*. For example

$$3y^2 - 2y + 4$$

is a polynomial in y.

Usually, we write a polynomial in either ascending powers of x or descending powers of x. Such a way of writing a polynomial is called a standard form of writing a polynomial. For example

$$4x^3 - 6x^2 + 2x + 1$$

is a polynominal in standard form with the powers written in descending order; and

$$2 - 3y + 6y^2 - 2y^3$$

is a polynomial in a standard form with the powers written in ascending order The polynomial

$$5u^3 - 6u + 1 - 2u^2$$

is not in a standard form, because the powers are neither in ascending nor in the descending order. This polynomial can be written in a standard form either as

$$5u^3 - 2u^2 - 6u + 1,$$

or as

$$1 - 6u - 2u^2 + 5u^3$$

A polynomial, having only one term, is called a *monomial* since 'mono' means one or single; a polynomial having two terms is called a *binomial* since 'bi' stands for two; and a polynomial having three terms is called a *trinomial* since 'tri' stands for three For example

$2, 3x, 7x^2$ are monomials,

$4-2x$; $6u^4 + 2u$; $y^8 - 2$ are binomials, and

$x^4 - 3x + 2$; $t^2 - 3t + \frac{2}{3}$ are trinomials.

The polynomial having all coefficients as zero is called *Zero Polynomial.* For example

$$0x^3 - 0x + 0,$$
$$0t^2 + 0t + 0,$$

are zero polynomials.

Exercises 4.1

1. Which of the following functions are polynomials?

   (*i*) $4x^2 - 3x + 2$ (*ii*) $\sqrt{2}y^3 + \sqrt{3}y$

   (*iii*) $\sqrt{2}\sqrt{t} + \sqrt{3}t$ (*iv*) $x + \frac{2}{x}$

   (*v*) $u^{-\frac{1}{2}} - 3u + 2$ (*vi*) $\frac{4}{3}x^7 - 3x^5 + 2x^3 - 1$

2. Write the following polynomials in a standard form:

   (*i*) $x^7 - 3x^5 + \sqrt{2}x + \frac{4}{3}x^2 - 2x^6 + 4$

   (*ii*) $x^5 + x^2 - \sqrt{3}x + 2x^4$

   (*iii*) $u^3 - u + u^2 - \sqrt{2}$

   (*iv*) $y^2 - 2y^4 + 3y - y^3 + 4$

3. In the following, identify the monomials, binomials and trinomials:

   (*i*) y^2

   (*ii*) $m^2 + 2m$

   (*iii*) $t^6 - \frac{5}{3}t^3 + 6$

   (*iv*) $7u^7 - 6u^6 + 5u^2$

(v) $x^6 - \sqrt{3x}$
(vi) $7u^6$
(vii) 7

4.2 Sum, Difference and Product of Two Polynomials

In elementary algebra the sum and difference of two algebraic expressions are found by grouping the like terms retaining their signs. Similarly, we find the sum and difference of two polynomials by grouping like powers, retaining their signs and adding the coefficients of like powers. For example, let

$$p(x) = x^4 - 3x^3 + 2x + 6,$$

and $q(x) = x^3 - 3x + 2$

Then
$$\begin{aligned} p(x) + q(x) &= (x^4 - 3x^3 + 2x + 6) + (x^3 - 3x + 2) \\ &= x^4 + (x^3 - 3x^3) + (2x - 3x) + (6 + 2) \\ &= x^4 - 2x^3 - x + 8, \end{aligned}$$

and
$$\begin{aligned} p(x) - q(x) &= (x^4 - 3x^3 + 2x + 6) - (x^3 - 3x + 2) \\ &= x^4 - 3x^3 + 2x + 6 - x^3 + 3x - 2 \\ &= x^4 + (-3x^3 - x^3) + (2x + 3x) + (6-2) \\ &= x^4 - 4x^3 + 5x + 4 \end{aligned}$$

For convenience the above working can be arranged as follows:

$$\begin{aligned} p(x) &= x^4 - 3x^3 + 2x + 6 \\ q(x) &= \underline{\quad\quad\ \ x^3 - 3x + 2} \\ \therefore\ p(x) + q(x) &= \underline{x^4 - 2x^3 - x + 8} \end{aligned}$$

Again
$$\begin{aligned} p(x) &= x^4 - 3x^3 + 2x + 6 \\ -q(x) &= \underline{\quad\quad -(x^3 + 3x - 2\,)} \\ p(x) - q(x) &= \underline{x^4 - 4x^3 + 5x + 4} \end{aligned}$$

Example 4.1: Find the sum of the polynomials

$4u^3 - 3u^2 + 2$ and $u^4 - 2u^3 + 6$

Solution: $(4u^3 - 3u^2 + 2) + (u^4 - 2u^3 + 6)$
$= u^4 + (4u^3 - 2u^3) - 3u^2 + (2 + 6)$
$= u^4 + 3u^3 - 3u^2 + 8$

Example 4.2: If $p(y) = 3y^7 - 2y^2 + 3$ and $q(y) = y^6 - 3y^4 + y^2 + y$, find $p(y) - q(y)$.

Solution: $p(y) - q(y) = (3y^7 - 2y^2 + 3) - (y^6 - 3y^4 + y^2 + y)$
$= 3y^7 - 2y^2 + 3 - y^6 + 3y^4 - y^2 - y$
$= 3y^7 - y^6 + 3y^4 + (-2y^2 - y^2) - y + 3.$
$= 3y^7 - y^6 + 3y^4 - 3y^2 - y + 3$

The product of two polynomials is found by using the distributive law for the product of algebraic expressions, and then grouping the like powers to add or subtract as the case may be.

Example 4.3: Find the product of $x^2 - 3x + 2$ and $x^3 - 6x^2 + x + 1$.

Solution: $(x^2 - 3x + 2)(x^3 - 6x^2 + x + 1)$

$$\begin{aligned}
&= x^2(x^3 - 6x^2 + x + 1) - 3x(x^3 - 6x^2 + x + 1) \\
&\quad + 2(x^3 - 6x^2 + x + 1) \\
&= (x^5 - 6x^4 + x^3 + x^2) - (3x^4 - 18x^3 + 3x^2 + 3x) \\
&\quad + (2x^3 - 12x^2 + 2x + 2) \\
&= x^5 + (-6 - 3)x^4 + (1 + 18 + 2)x^3 + (1 - 3 - 12)x^2 + \\
&\quad (-3 + 2)x + 2 \\
&= x^5 - 9x^4 + 21x^3 - 14x^2 - x + 2
\end{aligned}$$

Remark: The solution above explains the method. However it is more convenient to work out as follows:

$$\begin{array}{r}
x^3 - 6x^2 + x + 1 \qquad\qquad\quad \\
x^2 - 3x + 2 \qquad\qquad\quad \\
\hline
x^5 - 6x^4 + x^3 + x^2 \qquad\qquad\qquad\quad \\
-3x^4 + 18x^3 - 3x^2 - 3x \qquad\quad \\
+2x^3 - 12x^2 + 2x + 2 \\
\hline
x^5 - 9x^4 + 21x^3 - 14x^2 - x + 2 \\
\hline
\end{array}$$

Example 4.4: If $p(x) = 5x^2 - 6x + 1$ and $q(x) = 0$ find $p(x) + q(x)$; $p(x) - q(x)$; and $p(x)\,q(x)$

Solution: We have

$$\begin{aligned}
p(x) + q(x) &= (5x^2 - 6x + 1) + (0) \\
&= 5x^2 - 6x + 1 = p(x) \\
p(x) - q(x) &= (5x^2 - 6x + 1) - (0) \\
&= 5x^2 - 6x + 1 = p(x) \\
p(x)\,q(x) &= (5x^2 - 6x + 1)(0) \\
&= (5x^2)(0) - (6x)(0) + (1)(0) \\
&= 0
\end{aligned}$$

In Examples 4.1, 4.2, 4.3 and 4.4, we considered the sum, the difference and the product of two polynomials and we found that the result, in each case was a polynomial This is always true, The sum, difference and product of two polynomials are always polynomials. It is clear from example 4.4, that when a polynomial $p(x)$ is added to the zero polynomial, we get $p(x)$ itself. If the zero polynomial is subtracted from any polynomial $p(x)$, we get $p(x)$ itself, and if any polynomial is multiplied by the zero polynomial, we get the zero polynomial.

Degree of a Polynomial

Look at the polynomial $x^7 - 3x^5 + 4$. What is the term with the highest power of x? It is x^7 The exponent in this term is 7 Similarly in the polynomial $5y^2 - 3y + 2$ the term with the highest power is $5y^2$ and the exponent in this term is 2 We call the exponent in the term with the highest power as the *degree of the polynomial.* Thus the degree of $x^7 - 3x^5 + 4$ is 7; the degree of $5y^2 - 3y + 2$ is 2.

Example 4.5: Find the degree of the polynomials given below:

(a) $x^4 - 3x^2 + 1$ (b) $1 - 2y + 3y^6$

(c) 7

Solution: (a) The highest power term is x^4, and the exponent is 4. So the degree is 4

(b) The highest power term is $3y^6$, and the exponent is 6. So the degree is 6.

(c) The highest power term is 7 which can be written as $7x^0$ and so the exponent is 0. So the degree is 0.

We do not assign any degree to the zero polynomial. (Why?).

Example 4.6: Let $p(y) = y^3 - y^2 + 2$, and

$q(y) = y + 1$

Find the degrees of $p(y) + q(y)$ and $p(y) - q(y)$

Solution: $p(y) + q(y) = (y^3 - y^2 + 2) + (y + 1)$

$= (y^3 - y^2 + y + 3)$

Therefore, degree of $p(y) + q(y)$ is 3.

$$p(y) - q(y) = (y^3 - y^2 + 2) - (y + 1)$$
$$= y^3 - y^2 - y + 1$$

Therefore, the degree of $p(y) - q(y)$ is 3.

Example 4.7: Let $p(u) = u^7 - u^5 + 2u^2 + 1$, and $q(u) = -u^7 + u - 2$

Find the degree of $p(u) + q(u)$

Solution: $p(u) + q(u) = (u^7 - u^5 + 2u^2 + 1) + (-u^7 + u - 2)$

$= -u^5 + 2u^2 + u - 1$

Therefore, the degree of $p(u) + q(u) = 5$.

Example 4.8: If $p(t) = 3t^2 - 2t + 1$, $q(t) = 3t^2 + t - 1$,

find the degree of $p(t) - q(t)$

Solution: $p(t) - q(t) = (3t^2 - 2t + 1) - (3t^2 + t - 1)$

$= -3t + 2$,

and therefore, the degree of $p(t) - q(t)$ is 1.

From Examples 4.6, 4.7 and 4.8, we observe that the degree of the sum or difference of two polynomials is less than or equal to the degree of the polynomial with higher degree.

Example 4.9: If $p(x) = x^2 - 2x + 1$ and $q(x) = x^3 - 3x^2 + 2x - 1$

find the degree of $p(x)\, q(x)$.

Solution: $p(x)\, q(x) = (x^2 - 2x + 1)\,(x3 - 3x^2 + 2x - 1)$

$$= x^2 (x^3 - 3x^2 + 2x - 1) - 2x (x^3 - 3x^2 + 2x - 1) + 1 (x^3 - 3x^2 + 2x - 1),$$
$$= (x^5 - 3x^4 + 2x^3 - x^2 - (2x^4 - 6x^3 + 4x^2 - 2x) + (x^3 - 3x^2 + 2x - 1)$$
$$= x^5 - 5x^4 + 9x^3 - 8x^2 + 4x - 1.$$

So the degree of $p(x)\,q(x)$ is 5.

In the above examples, the degree of $p(x)$ is 2, the degree of $q(x)$ is 3 and the degree of $p(x)\,q(x)$ is 5 We see that degree of $p(x)$ + degree $q(x)$ = degree $p(x)\,q(x)$

In general, we can show that the degree of the product of two non-zero polynomials is the sum of the degrees of the factors in the product.

Exercises 4.2

1. Find the degree of each of the following polynomials

   (i) $3x + 5$ (ii) 3

   (iii) 5 (iv) 0

   (v) $2y + 3 - 5y^2$ (vi) $3m^2 - 5m + 7m^4$

   (vii) $u^3(2 + 3u)$ (viii) $(2t + 1)(5t - 7)$

2. Find the sums of the following groups of polynomials and also find the degree of the sum in each case.

   (a) $x^3 - 5x^2 + x + 2$ and $x^3 - 3x^2 + 2x + 1$

   (b) $3x^2 + 5x - 2$ and $-3x^2 - 5x + 6$

   (c) $y^6 - 3y^4$ and $y^4 + y^3 + 2y^2 - 6$

   (d) $t^2 + t - 7$ and $t^2 + t^2 + 3t + 4$

   (e) $3u^2 - 3u + 6$, $-u^2 + 4u + 3$ and $-2u^2 + 4$

3. In the following subtract the second polynomial from the first and find the degree of this difference:

   (a) $x^3 - 3x^2 + 6$; $x^2 - x + 4$

   (b) $u^7 - 3u^6 + 4u^2 + 2$; $u^6 - u - 4$

   (c) $y^3 - 3y^2 + y + 2$; $y^3 + 2y + 1$

   (d) $t^4 - 3t^3 + 2t + 6$; $t^4 - 3t^3 - 6t + 2$

4. From the sum of $u^4 - 3u^3 + 2u + 6$ and $u^4 - 3u^2 + 6u + 2$ subtract $u^3 - 3u + 4$.

5. What should be added to $x^4 - x^2 + x + 2$ to get $x^2 + x + 4$?

6. What should be subtracted from $x^3 - 2x^2 + 4x + 1$ to get 1?

7. In the following find the product $p(x)\,q(x)$ and find the degree of this product:

   (i) $p(x) = x + 3$ and $q(x) = x - 2$

   (ii) $p(x) = x^2 - 4x + 4$ and $q(x) = x - 2$

   (iii) $p(x) = x^2 + 3x + 2$ and $q(x) = x^2 + 3x + 1$

8. Find $p(u)\,q(u)$ if $p(u) = u^2 + 3u + 1$ and $q(u) = u^3 - u^2 + 4$.

4.3 Division of a Polynomial by a Polynomial

Let us consider two numbers 28 and 3. We know how to divide 28 by 3. We get quotient 9 and remainder 1. We write 28 as

$$28 = 9 \times 3 + 1$$

We observe that the remainder 1 is less than 3, the divisor.

Similarly we get
$27 = 9 \times 3 + 0$
Here the remainder is 0 and we say that 3 is a factor of 27 Also observe that $3 \neq 0$, that is, the divisor is not zero. (Remember· division by zero is not defined).

Now suppose $p(x)$ and $g(x)$ are two polynomials and $g(x) \neq 0$ If we can find polynomials $q(x)$ and $r(x)$, such that:

$p(x) = g(x)\, q(x) + r(x)$,

where $r(x) = 0$ or degree $r(x) <$ degree $g(x)$, then we say that $p(x)$ divided by $g(x)$, gives $q(x)$ as quotient and $r(x)$ as remainder.

If the remainder $r(x)$ is zero, we say that divisor $g(x)$ is a factor of $p(x)$.

We explain, by means of the following examples, the method of dividing a polynomial $p(x)$ by another polynomial $g(x)$ of smaller degree

Example 4.10: Divide $p(x)$ by $g(x)$, where
$p(x) = x^4 + 1$ and $g(x) = x+1$

Solution:

$$
\begin{array}{r|r}
 & x^3 - x^2 + x - 1 \\
\hline
x + 1 & x^4 + 1 \\
 & -(x^4 + x^3) \\
\hline
 & -x^3 + 1 \\
 & -(-x^3 - x^2) \\
\hline
 & x^2 + 1 \\
 & -(x^2 + x) \\
\hline
 & -x + 1 \\
 & -(-x - 1) \\
\hline
 & 2
\end{array}
$$

Here, the quotient $q(x) = x^3 - x^2 + x - 1$ and the remainder $r(x) = 2$.
We write $x^4 + 1 = (x^3 - x^2 + x - 1)(x + 1) + 2$

Note: Notice that the degree of $g(x)$ is less than the degree of $p(x)$. Therefore it is always possible to divide a polynomial of higher degree by a polynomial of lower degree (in the same variable). The process above stops as soon as the remainder is zero or the degree of the remainder becomes smaller than that of the divisor.

In the above example, the remainder $r(x)$ $(=2)$ is not zero and so the divisor $x + 1$ is not a factor of $p(x) = x^4 + 1$.

Example 4.11: Divide $p(y)$ by $g(y)$, if $p(y) = y^3 - 3y^2 - y + 3$ and $g(y) = y^2 - 4y + 3$

Solution:

$$
\begin{array}{r|l}
 & \quad y + 1 \\
\hline
y^2 - 4y + 3 & y^3 - 3y^2 - y + 3 \\
 & -(y^3 - 4y^2 + 3y) \\
\cline{2-2}
 & \qquad y^2 - 4y + 3 \\
 & \qquad -(y^2 - 4y + 3) \\
\cline{2-2}
 & \qquad\qquad 0
\end{array}
$$

Here the quotient is $y+1$ and the remainder is zero. We write
$y^3 - 3y^2 - y + 3 = (y + 1)(y^2 - 4y + 3)$

Example 4.12: Examine if $y - 3$ is a factor of $p(y) = y^3 - 2y^2 + 3y - 18$

Solution: If $y-3$ is a factor of $p(y)$ the remainder must be 0, when $p(y)$ is divided by $y-3$ Let us see if the remainder is zero by performing the division.

$$
\begin{array}{r|l}
 & \quad y^2 + y + 6 \\
\hline
y-3 & y^3 - 2y^2 + 3y - 18 \\
 & -(y^3 - 3y^2) \\
\cline{2-2}
 & \qquad y^2 + 3y - 18 \\
 & \quad -(y^2 - 3y) \\
\cline{2-2}
 & \qquad\quad 6y - 18 \\
 & \qquad -(6y - 18) \\
\cline{2-2}
 & \qquad\qquad 0
\end{array}
$$

Since the remainder is zero, $(y-3)$ is a factor of $y^3 - 2y^2 + 3y - 18$.

Exercises 4.3

In each of the following questions divide the polynomial p by g and find the quotient and the remainder. Find in which cases g is a factor of p

1. $p(x) = 5x^2 + 3x + 1$; $g(x) = 2x$
2. $p(y) = y^3 - 3y^2 + 4y + 2$; $g(y) = y-1$
3. $p(u) = u^3 + 3u^2 - 12u + 4$, $g(u) = u-2$
4. $p(x) = x^3 - 14x^2 + 37x - 60$, $g(x) = x-2$
5. $p(y) = y^6 + 3y^2 + 10$, $g(y) = y^3 + 1$
6. $p(t) = t^3 - 6t^2 + 11t - 6$, $g(t) = t^2 - 5t + 6$
7. $p(x) = x^5 + 5x^3 + 3x^2 + 5x + 3$, $g(x) = x^2 + 4x + 2$
8. $p(u) = u^5 + u^4 + u^3 + u^2 + 2u + 2$; $g(u) = u^3 + 1$
9. $p(x) = 2x^2 - 3x + 5$, $g(x) = x-a$

4.4 Remainder Theorem

We will now explain a method for finding the remainder, without actually performing the process of division, when a polynomial $p(x)$ with degree greater than one, is divided by a binomial of the form $x-a$. We recall that the degree of the remainder should be either less than the degree of the divisor or the remainder must be zero. Since divisor is $x-a$ and has degree 1, the remainder must be zero or have degree zero, that is, it must be a constant which may also be zero. In any case, the remainder must be a constant. Let us consider some examples.

Example 4.13 Let $p(x) = x^4 + 2x^3 - 3x^2 + x - 1$.
Find the remainder when $p(x)$ is divided by $x-2$

Solution:

$$\begin{array}{r|l}
 & x^3 + 4x^2 + 5x + 11 \\ \hline
x-2 & x^4 + 2x^3 - 3x^2 + x - 1 \\
 & -(x^4 - 2x^3) \\ \hline
 & \quad 4x^3 - 3x^2 + x - 1 \\
 & \quad -(4x^3 - 8x^2) \\ \hline
 & \qquad 5x^2 + x - 1 \\
 & \qquad -(5x^2 - 10x) \\ \hline
 & \qquad\quad 11x - 1 \\
 & \qquad\quad -(11x - 22) \\ \hline
 & \qquad\qquad 21
\end{array}$$

Remainder is 21.

In the above example let us evaluate $p(2)$. We get

$$\begin{aligned} p(2) &= 2^4 + 2(2^3) - 3(2^2) + 2 - 1 \\ &= 16 + 16 - 12 + 2 - 1 \\ &= 21. \end{aligned}$$

Thus we find that $p(2)$ is equal to the remainder when $p(x)$, is divided by $x-2$. Let us consider another example.

Example 4.14: Find the remainder when $y^3 + y^2 - 2y + 1$ is divided by $y - 3$.

Here $p(y) = y^3 + y^2 - 2y + 1$

Solution:

$$\begin{array}{r|l}
 & y^2 + 4y + 10 \\ \hline
y-3 & y^3 + y^2 - 2y + 1 \\
 & -(y^3 - 3y^2) \\ \hline
 & \quad 4y^2 - 2y + 1 \\
 & \quad -(4y^2 - 12y) \\ \hline
 & \qquad 10y + 1 \\
 & \qquad -(10y - 30) \\ \hline
 & \qquad\quad 31
\end{array}$$

Here remainder is 31. What is $p(3)$? We have

$$\begin{aligned} p(3) &= 3^3 + 3^2 - 2.3 + 1 \\ &= 27 + 9 - 6 + 1 \\ &= 31 \end{aligned}$$

Thus we find that $p(3)$ is the remainder when $p(y)$ is divided by $x-3$.

Example 4.15: Let $p(y) = y^3 + y^2 - 2y + 1$ Find the remainder when $p(y)$ is divided by $y-a$. Also find $p(a)$. Do you see any relation between the remainder and $p(a)$?

Solution:

$$\begin{array}{r|l} & \quad y^2 + (1+a)y + a(1+a)-2 \\ \hline y-a & \quad y^3 + y^2 - 2y + 1 \\ & -(y^3 - ay^2) \\ \hline & \quad (1+a)y^2 - 2y + 1 \\ & \quad -[(1+a)y^2 - a(1+a)y] \\ \hline & \qquad [a(1+a)-2]\,y + 1 \\ & \qquad -[a(1+a)-2]y - a[a\,(1+a)-2] \\ \hline & \qquad\qquad 1 + a[a(1+a)-2] \end{array}$$

Here the remainder is

$$\begin{aligned} & 1 + a[a(1+a) - 2] \\ &= 1 + a[a^2 + a - 2] \\ &= a^3 + a^2 - 2a + 1 \end{aligned}$$

Also $p(a) = a^3 + a^2 - 2a + 1$

So we find that the remainder is equal to $p(a)$

You may have observed that examples 4.13, 4.14, and 4.15 exhibit a relationship between $p(a)$, the value of $p(x)$ for $x = a$, and the remainder obtained when $p(x)$ is divided by $x-a$. Actually the remainder $= p(a)$.

We state this result in the form of a theorem.

Theorem 4.1: Remainder Theorem.

Let $p(x)$ be any polynomial of degree ≥ 1, and a any real number. If $p(x)$ is divided by $x - a$, then the remainder is $p(a)$.

The remainder theorem can be proved as follows.

Proof: Let us suppose that, when $p(x)$ is divided by $x-a$, the quotient is $q(x)$ and remainder is $r(x)$. So we have

$$p(x) = (x-a)\,q(x) + r(x), \text{where } r(x) = 0 \text{ or degree } r(x) < \text{degree } (x-a)$$

Since degree of $(x-a)$ is 1, either $r(x) = 0$ or degree of $r(x) = 0\ (< 1)$. So $r(x)$ is a constant, say r.

Example 4.16: Determine the remainder when the polynomial $p(x) = x^4 - 3x^2 + 2x + 1$ is divided by $x - 1$.

Solution: By the remainder theorem, the required remainder $= p(1)$

$= 1^4 - 3(1^2) + 2(1) + 1$

$= 1 - 3 + 2 + 1$

$= 1$

Example 4.17: Find the remainder when $p(x) = x^2 + 4x + 2$ is divided by $x + 2$.

Solution: $x + 2 = x - (-2)$, so here $a = -2$.

$\therefore$ the remainder $= p(a) = p(-2)$

$= (-2)^2 + 4(-2) + 2$

$= 4 - 8 + 2$

$= -2$

Example 4.18: Determine whether $x - 3$ is a factor of $p(x) = x^3 - 3x^2 + 4x - 12$

Solution: $p(3) = 3^3 - 3(3^2) + 4(3) - 12$

$= 27 - 27 + 12 - 12$

$= 0$

Since $p(3) = 0$, $x - 3$ is a factor of $p(x)$ by the factor theorem

Example 4.19: Find the value of a if $x - a$ is a factor of $x^3 - a^2x + x + 2$

Solution: Since $x - a$ is a factor of $p(x) = x^3 - a^2x + x + 2$, we must have $p(a) = 0$.

$\therefore a^3 - a^2(a) + a + 2 = 0$

i.e $a + 2 = 0$,

$\therefore a = -2$

Exercises 4.4

1 If $p(x) = 4x^3 - 3x^2 + 2x - 4$, find the remainder when $p(x)$ is divided by

(i) $x - 1$ (ii) $x - 2$ (iii) $x + 1$

(iv) $x - 4$ (v) $x + 2$ (vi) $x + \frac{1}{2}$

2. Use the factor theorem to determine if $x - 1$ is a factor of $x^6 - x^5 + x^4 - x^3 + x^2 - x + 1$.

3. In the following problems use the factor theorem to find if $g(x)$ is a factor of $p(x)$:

(i) $p(x) = x^3 - 3x^2 + 4x - 4$ and $g(x) = x - 2$

(ii) $p(x) = 2x^3 + 4x + 6$, and $g(x) = x + 1$

(iii) $p(x) = x^3 + x^2 + 3x + 175$ and $g(x) = x + 5$

4. If $x - 2$ is a factor of each of the following three polynomials, find the value of a in each case:

(i) $x^2 - 3x + 5a$ (ii) $x^3 - 2ax^2 + ax - 1$

(iii) $x^5 - 3x^4 - ax^3 + 3ax^2 + 2ax + 4$

5. In each the following two polynomials, find the value of a if $x + a$ is a factor

(i) $x^3 + ax^2 - 2x + a + 4$,
(ii) $x^4 - a^2x^2 + 3x - a$

6. In each the following two polynomials, find the value of a if $x - a$ is a factor

(i) $x^6 - ax^5 + x^4 - ax^3 + 3x - a + 2$
(ii) $x^5 - a^2x^3 + 2x + a + 1$

CHAPTER 5

Factoring a Polynomial

5.1 Introduction

So far as the properties of addition, multiplication,subtraction and division are concerned, polynomials behave in exactly the same way as the integers do.

The product of two integers, say 18 and 15, is another integer 270. We write

$$18 \times 15 = 270$$

Viewed the other way about, 18 and 15 are two factors of 270, and we write

$$270 = 18 \times 15$$

Now, 270 has also factors other than 18 and 15. As such, 270 can be expressed as a product of its factors in more than one way. For example,

$$270 = 6 \times 3 \times 15, \quad 270 = 2 \times 3 \times 3 \times 15, \quad 270 = 18 \times 5 \times 3,$$
$$270 = 6 \times 3 \times 5 \times 3, \quad 270 = 2 \times 3 \times 3 \times 3 \times 5$$

There is something special about the last expression. Each of the factors is a prime positive integer. We have 270 also a positive integer. And this is the only way the positive integer 270 can be expressed as a product of its prime factors.

In exactly the same way, if $g(x)$ and $h(x)$ are two polynomials, their product is a polynomial $p(x)$. We write

$$g(x)\, h(x) = p(x)$$

Viewed the other way about, $g(x)$ and $h(x)$ are two factors of $p(x)$, and we write

$$p(x) = g(x)\, h(x)$$

Thus, $p(x)$ can be factored (or factorised) into two factors $g(x)$ and $h(x)$. Now, $p(x)$ may admit of factors other than $g(x)$ and $h(x)$ as well, and in that case it may be possible to express $p(x)$ as a product of its factors in more than one way.

Perhaps, by now, you may have begun to wonder: (*i*) whether there are polynomials which may be called prime polynomials, (*ii*) whether it is possible to express every other (non-prime) polynomial as a product of its prime factors, and (*iii*) whether there is exactly one way in which a polynomial may be expressed as a product of its prime factors.

You will discover the answer to these questions gradually.

Some Special Names of Polynomials

Consider the following polynomials:

(1) $x + 3$ (2) $2x - 7$

(3) $x^2 - 6x + 2$ (4) $2x^2 - 3x + 1$

(5) $x^3 - 3x^2 + 4$ (6) $3x^3 - 2x^2 + 6x - 1$

(7) $8x^4 - 6x^3 + 3x^2 + x - 3$.

In these, (1) and (2) are polynomials of degree 1, (3) and (4) are of degree 2, (5) and (6) are of degree 3; and (7) is of degree 4

A polynomial of degree 1 is called a *linear polynomial*, a polynomial of degree 2 is called a *quadratic polynomial*, a polynomial of degree 3 is called a *cubic polynomial*, a polynomial of degree 4 is called a *biquadratic polynomial* (or a *quartic polynomial*); and so on

A linear polynomial in x is in general of the form $ax + b$ and a quadratic polynomial in x is in general of the form $ax^2 + bx + c$; In both these polynomials, $a \neq 0$.

Methods of Factoring A Polynomial

There are no standard methods or procedures for finding the factors of a polynomial, except in some very special cases.

If c is a non-zero real number, so that c is a (constant) polynomial of degree 0, we have for any polynomial $f(x)$

$$f(x) = c\,[\frac{1}{c} f(x)\,]$$

which shows that c is a factor of $f(x)$. Thus a non-zero constant c is a factor of every polynomial. Therefore, the problem of finding factors of a polynomial, in practice, reduces to that of finding factors of positive degree, that is, of degree $n \geq 1$

In what follows, we shall attempt to find, where possible, linear factors of a quadratic polynomial

$$ax^2 + bx + c,\ a, b, c \in R,\ a \neq 0.$$

We emphasise the phrase "where possible". It can be easily shown by an example that not every quadratic polynomial can be factored into linear factors. Consider for example

$$4x^2 - 4x + 2$$

For any real number h, $4h^2 - 4h + 2 = (2h - 1)^2 + 1 > 0$ and therefore $4h^2 - 4h + 2 \neq 0$ and hence $(x - h)$ is not a factor of $4x^2 - 4x + 2$. Consequently $4x^2 - 4x + 2$ cannot be factored into linear factors. However, for the present we shall consider only such quadratic polynomials as can be factored into linear factors.

5.2 Factoring a Quadratic Polynomial

Recall the following formulae, you learnt in your lower classes·

(i) $(x + a)^2 = x^2 + 2ax + a^2$

(ii) $(x - a)^2 = x^2 - 2ax + a^2$

(iii) $(x - a)(x + a) = x^2 - a^2$

Each of the above formulae has two interpretations. For example,
(i) states

$$(x + a)^2 = x^2 + 2ax + a^2$$

Viewed as a multiplication problem, we know that when $(x + a)$ is multiplied by itself, the product can be written as $x^2 + 2ax + a^2$, a quadratic polynomial. Viewed as a problem of factorisation, we find that a quadratic polynomial of the form $x^2 + 2ax + a^2$ can be factorised as $(x + a)(x + a)$, which in short, can be written as $(x + a)^2$

Similar remarks apply to the other two formulae. Hence, if a quadratic polynomial fits the expression on the right-hand side, in one of the above formulae, the factor, can be immediately written down, as on the left-hand side

Example 5.1: Factorise: $x^2 - 18x + 81$

Solution: $x^2 - 18x + 81 = x^2 - 18x + 9^2$
$= x^2 - 2.9x + 9^2$
$= (x - 9)^2$

Example 5.2: Resolve into factors: $49x^2 + 14x + 1$

Solution: $49x^2 + 14x + 1 = (7x)^2 + 2(7x)\ 1 + 1^2$
$= (7x + 1)^2$

Example 5.3: Factorise: $45\,a^3b + 5ab^3 - 30a^2b^2$

Solution: $45\,a^3b + 5\,ab^3 - 30\,a^2b^2 = 5\,ab\,(9a^2 + b^2 - 6ab)$
$= 5\,ab\,[(3a)^2 - 2\,(3a)b + b^2\,]$
$= 5\,ab\,(3a - b)^2$

Remarks: It was easy to observe that the monomial 5 *ab* was a factor common to all the terms. So we used distributive law. The expression within brackets is easily recognised as a polynomial in *a*.

Example 5.4: Factorise: $49 - 64x^2$

Solution: $49 - 64x^2 = (7)^2 - (8x)^2$
$= (7 - 8x)(7 + 8x)$

Example 5.5: Factorise: $16\,(2x - 1)^2 - 25z^2$

Solution: $16\,(2x - 1)^2 - 25z^2 = [4\,(2x - 1)\,]^2 - (5z)^2$
$= [4\,(2x - 1) - 5z]\,[4\,(2x - 1) + 5z\,]$
$= (8x - 5z - 4)(8x + 5z - 4)$

Example 5.6: Resolve into factors: $25x^2 - 10x + 1 - 36\,y^2$

Solution: $25\,x^2 - 10x + 1 - 36\,y^2 = (25x^2 - 10x + 1\,) - (6y)^2$
$= [\,(5x)^2 - 2(5x).1 + 1^2\,] - (6y)^2$
$= (5x - 1)^2 - (6y)^2$
$= (5x - 1 - 6y)(5x - 1 + 6y)$

Exercises 5.1

1. Resolve into factors:

(*i*) $4x^2 + 12x + 9$	(*ii*) $9x^2 - 24x + 16$
(*iii*) $144y^2 + 24y + 1$	(*iv*) $36x^2 + 25 + 60x$
(*v*) $100 - 9z^2$	(*vi*) $49 - 64k^2$
(*vii*) $a^2 - 25b^2$	(*viii*) $4a^2 - (2b - c)^2$
(*ix*) $18x^2a^2 - 32$	(*x*) $3x^3y - 243xy^3$

2. Factorise:

(*i*) $x^2 - y^2 + 6y - 9$	(*ii*) $25x^2 - 10x + 1 - 36z^2$
(*iii*) $x^3 - x$	(*iv*) $1 - 2ab + (a^2 + b^2)$

Double (*v*) $4(x - y)^2 - 12(x - y)(x + y) + 9(x + y)^2$

Double (*vi*) $a^2 + b^2 + 2(ab + bc + ca)$

5.3 Method of Splitting the Middle Term

We want to factorise $x^2 + bx + c$ into factors of the form $x + p$ and $x + q$.

Suppose that $x^2 + bx + c = (x + p)(x + q)$

$= x^2 + (p + q)x + pq$ identically.

Comparing both the sides we get

$p + q = b,$

and $pq = c.$

Thus, if we find p and q satisfying the above two conditions, then we can write

$x^2 + bx + c$

$= x^2 + px + qx + pq$

$= (x^2 + px) + (qx + pq$

$= x(x + p) + q(x + p)$

$= (x + q)(x + p)$

Thus in order to factorise $x^2 + bx + c$ we have to find numbers p and q such that their sum is b and their product is c. Once we find p and q, we split the middle term bx in the quadratic as $px + qx$ and get the desired factorisation by grouping the terms, as shown above We shall now explain this method by means of examples.

Examples 5.7: Factorise $x^2 + 6x + 8$.

Solution: We want to find p, q such that $p + q = 6$ and $pq = 8$

So p, q may be two factors of 8 such that their sum is 6. Let us examine factors of 8. We have

$8 = 1 \times 8$, but the sum of the two factors $= 1 + 8 = 9 \neq 6$

Also $8 = 2 \times 4$; here the sum of the two factors $= 2 + 4 = 6$, as we wanted. So we take $p = 2$; $q = 4$.

We now split middle term $6x$ in the quadratic as $2x + 4x$, so that:

$$x^2 + 6x + 8 = x^2 + 2x + 4x + 8$$
$$= (x^2 + 2x) + (4x + 8)$$

$$= x(x + 2) + 4(x + 2)$$
$$= (x + 2)(x + 4)$$

Thus, $x^2 + 6x + 8 = (x + 2)(x + 4)$

Example 5.8: Factorise $x^2 + 5x - 24$.

Solution: We need p, q such that

$$pq = -24; p + q = 5$$

Now p and q may be factors of -24 such that their sum is 5.

Factors of -24 are $(1) \times (-24)$,
$(-1) \times 24$,
$2 \times (-12)$,
$(-2) \times 12$,
$3 \times (-8)$,
$(-3) \times 8$, etc

If we take $p = -3, q = 8$, then $p + q = 5$ as required. Thus we split the middle term as follows:

$$x^2 + 5x - 24 = x^2 - 3x + 8x - 24$$
$$= (x^2 - 3x) + (8x - 24)$$
$$= x(x - 3) + 8(x - 3)$$
$$= (x - 3)(x + 8)$$

Hence the required factorisation is

$$x^2 + 5x - 24 = (x - 3)(x + 8)$$

Example 5.9: Factorise $x^2 - 5x + 6$.

Solution: In this case $pq = 6; p + q = -5$

Factors of 6: 1×6,
$(-1) \times (-6)$,
2×3,
and $(-2) \times (-3)$

we take $p = -2; q = -3$, then $pq = 6, p + q = -5$

Now, by splitting the middle term, we get

$$x^2 - 5x + 6 = x^2 - 2x - 3x + 6$$
$$= (x^2 - 2x) + (-3x + 6)$$
$$= x(x - 2) - 3(x - 2)$$
$$= (x - 2)(x - 3)$$
$$\therefore x^2 - 5x + 6 = (x - 2)(x - 3)$$

Example 5.10: Factorise $x^2 - 5x - 14$.

Solution: Here $pq = -14, p + q = -5$

Factors of -14 : $(-1) \times 14$,
$1 \times (-14)$,
$2 \times (-7)$,
and $(-2) \times (7)$

If $p = 2$ and $q = -7$, then $pq = -14$ and $p + q = -5$

Now we split middle term to get

$$\begin{aligned} x^2 - 5x - 14 &= x^2 + 2x - 7x - 14 \\ &= (x^2 + 2x) - (7x + 14) \\ &= x(x + 2) - 7(x + 2) \\ &= (x + 2)(x - 7) \\ \therefore x^2 - 5x - 14 &= (x + 2)(x - 7) \end{aligned}$$

Example 5.11: Consider $x^2 + x + 1$. Can we factorise this ? We need p, q such that $pq = 1$ and $p + q = 1$ It is not possible to find any such p and q (Why ?) Thus this quadratic cannot be factorised. As we mentioned in the introduction, you will learn the condition under which a quadratic can be factorised, in your higher classes

Exercises 5.2

1. Factorise the following quadratic polynomials by splitting the middle term·

(i) $x^2 + 14x + 45$	(ii) $x^2 - 22x + 120$
(iii) $x^2 - 11x - 42$	(iv) $x^2 - 21x + 108$
(v) $u^2 - 30u + 216$	(vi) $y^2 + 2y - 3$
(vii) $x^2 + 14x + 48$	(viii) $y^2 - 4x - 21$
(ix) $x^2 + 5x - 36$	(x) $x^2 - 23x + 132$

5.4 Method of Splitting the Middle Term (Continued)

We shall now see how we can use the method of splitting the middle term to factorise quadratic polynomials of the form $ax^2 + bx + c$. (In Section 5.3 we learnt this method when $a = 1$).

Evidently we cannot factorise $ax^2 + bx + c$ as $(x + p)(x + q)$ (when $a \neq 1$) because then the product of $(x + p)(x + q)$ will have 1 as the coefficient of x^2, whereas the coefficient is $a \neq 1$. So we try to factorise it as a product of linear polynomials

$(px + q)$ and $(rx + s)$

$$\begin{aligned} \text{Let } ax^2 + bx + c &= (px + q)(rx + s) \\ &= prx^2 + (ps + qr)x + qs \end{aligned}$$

Comparing the coefficients on both the sides, we get

$$\begin{aligned} a &= pr, \\ b &= ps + qr \\ \text{and } c &= qs \end{aligned}$$

Now $ac = (pr)(qs) = (ps)(qr)$

If we put $l = ps$ and $m = qr$, then we get

$$\begin{aligned} lm &= ac \\ l + m &= b, \end{aligned}$$

once again, splitting middle term bx as $(l + m)x$.

So we have to find l, m such that

$$lm = ac$$
$$l + m = b$$

After splitting the middle term, we then group terms as in Section 5.3 We explain this procedure in the following examples.

Example 5.12: Factorise $6x^2 - x - 2$

Solution: Here $a = 6, b = -1, c = -2$

So we want to find l, m such that

$$lm = ac = -12$$
$$l + m = b = -1$$

Factors of -12 . $1 \times (-12), (-1) \times 12; 2 \times (-6), (-2) \times 6.$

$3 \times (-4); (-3) \times 4$

We see that if we take $l = 3, m = -4.$

We get $lm = -12 = ac$

$l + m = -1 = b$

We now do the splitting of the middle term

$$\begin{aligned} 6x^2 - x - 2 &= 6x^2 + (3x - 4x) - 2 \\ &= (6x^2 + 3x) - (4x + 2) \\ &= 3x(2x + 1) - 2(2x + 1) \\ &= (2x + 1)(3x - 2) \end{aligned}$$

i.e., $6x^2 - x - 2 = (2x + 1)(3x - 2)$

Example 5.13: Factorise $5x^2 - 32x + 12$

Solution: $a = 5; b = -32, c = 12$

We want to find l, m such that

$$lm = 5 \times 12 = 60$$
$$l + m = -32$$

l, m are factors of 60 whose sum is -32

We have factors of 60. $1 \times 60, (-1) \times (-60)$

$2 \times 30; (-2) \times (-30).$

We see that if $l = -2, m = -30$ we get

$lm = 60; l + m = -32$

Thus we can split middle term $-32x$ as $-2x - 30x$ and get

$$\begin{aligned} 5x^2 - 32x + 12 &= 5x^2 - 2x - 30x + 12 \\ &= (5x^2 - 2x) - (30x - 12) \\ &= x(5x - 2) - 6(5x - 2) \\ &= (5x - 2)(x - 6) \end{aligned}$$

Thus, $5x^2 - 32x + 12 = (5x - 2)(x - 6)$

Exercises 5.3

Factorise the following by splitting the middle term:

(i) $3x^2 - 2x - 8$ (ii) $6x^2 - 5x - 6$

(iii) $3u^2 - 10u + 8$ (iv) $6u^2 + 17u + 12$

(v) $2y^2 - y - 21$ (vi) $12x^2 - 25x + 12$

(vii) $14x^2 + 19x - 3$ (viii) $30x^2 + 7x - 15$

(ix) $24x^2 - 65x + 21$ (x) $6x^2 - 16x - 21$

(xi) $\sqrt{3}y^2 + 11y + 6\sqrt{3}$ (xii) $\frac{1}{2}x^2 - 3x + 4$

(xiii) $4\sqrt{3}x^2 + 5x - 2\sqrt{3}$ (xiv) $14x^2 + 9x + 1$

(xv) $px^2 + (4p^2 - 3q)x - 12pq$

CHAPTER 6

Linear Equations in One Variable

6.1 Linear Equations

We recall that an equation is a statement of equality which involves one or more unknown quantities, called the "variables." We shall study in this chapter only equations involving one variable. Examine the following examples:

(*i*) $3x + 4 = 2$

(*ii*) $\frac{1}{2}y + 1 = 3y + \frac{2}{3}$

(*iii*) $2x + 3 = 4x + 1$

(*iv*) $x^2 + 4 = 2x$

(*v*) $x^3 + 4x^2 = 3x + 2$

In the examples (*i*), (*ii*) and (*iii*) above, only linear polynomials occur, whereas in (*iv*) and (*v*) quadratic and cubic polynomials are involved. An equation involving only linear polynomials is called a **Linear Equation.**

Exercises 6.1

Which of the following equations are linear equations?

(*i*) $\frac{3}{2}x + 4 = 2x - 3$ (*ii*) $5y - 3 = 25 + 4$

(*iii*) $u + 4 = u^2 - 4$ (*iv*) $3x + 2 = 5x - 7$

(*v*) $x^2 + 2 = x + 1$ (*vi*) $y - 3 = 3y + 4$

(*vii*) $u + \frac{1}{u} = 5$ (*viii*) $(x - 1)^2 = x^2 - 2$

(*ix*) $(x - 1)(x - 2) = 6$ (*x*) $(x - 2)(x + 3) = x + 7$

6.2 Solution of a Linear Equation

Consider the linear equation

$$3x - 2 = 7$$

If we substitute the value 2 for x, we get

Left-Hand Side (L.H.S) $= 6 - 2 = 4$,

Right-Hand Side (R.H.S) $= 7$

and therefore L.H S. $\neq$ R.H S.

If we substitute the value 3 for x, we get

$$\text{L.H.S.} = 9 - 2 = 7 = \text{R.H.S.}$$

A value of the variable, which, when substituted for the variable in the equation, makes the two sides (of the equation) equal, is called the "solution" of the equation. Thus 2 is not a solution of the above equation, whereas 3 is a solution.

Example 6.1: Verify that -3 is a solution of

$$3x + 2 = -7$$

Solution: If we substitute $x = -3$, we get

$$\text{L.H.S.} = -9 + 2 = -7 = \text{R.H.S.}$$

Therefore -3 is a solution.

Example 6.2: Verify that -2 is a solution of

$$2x + 6 = 5x + 12$$

Solution: If we substitute $x = -2$, we get

$$\text{L.H.S.} = -4 + 6 = 2$$

$$\text{and R.H.S.} = -10 + 12 = 2$$

Therefore L.H.S. = R.H.S. Hence, -2 is a solution of $2x + 6 = 5x + 12$.

Example 6.3: Examine whether $\frac{1}{2}$ is a solution of

$$3x + \frac{1}{2} = 5x - \frac{1}{2}$$

Solution: If we substitute $x = \frac{1}{2}$, we get

$$\text{L.H.S.} = \frac{3}{2} + \frac{1}{2} = 2$$

$$\text{and R.H.S.} = \frac{5}{2} - \frac{1}{2} = 2$$

So, L.H.S. = R.H.S. Thus $\frac{1}{2}$ is a solution of the equation

Example 6.4: Examine whether 5 is a solution of

$$2x + 6 = 3x - 8$$

Solution: If we substitute $x = 5$, we get

$$\text{L.H.S.} = 10 + 6 = 16$$

$$\text{and R.H.S} = 15 - 8 = 7$$

So, L.H.S. $\neq$ R.H.S. Therefore 5 is not a solution of this equation

Solving a linear equation means finding a value of the variable which satisfies the equation. How does one solve a linear equation? We use the following obvious properties of equality:

(1) We can add the same quantity to both sides of an equality without changing the equality

(2) We can subtract the same quantity from both sides of an equality without changing it.

(3) We can multiply both sides of an equality by the same non-zero number without changing it.

(4) We can divide both sides of an equality by the same non-zero number without changing it.

We now solve the following equation using the above properties.

Example 6.5: Solve $2x + 3 = 7$

Solution: Adding -3 to both sides, we get

$$2x + 3 + (-3) = 7 + (-3), \text{ which gives}$$
$$2x = 4$$

Dividing both sides by 2, we get

$$\frac{1}{2}(2x) = \frac{1}{2}(4), \text{ which gives } x = 2$$

Thus 2 is the solution.

Example 6.6: Solve $12x + 4 = 4x + 28$

Solution: Add -4 to both sides to get

$$12x + 4 + (-4) = 4x + 28 + (-4)$$
i.e., $12x = 4x + 24$

Add $-4x$ to both sides to get

$$12x - 4x = 4x + 24 - 4x \text{ which gives}$$
$$8x = 24$$

Dividing both sides by 8, we get

$$\frac{1}{8}(8x) = \frac{1}{8}(24), \text{ giving}$$
$$x = 3$$

Thus 3 is the solution.

Note: In Example 6.5 above, instead of adding -3 to both the sides of the equation, transposing 3 to R.H.S. after changing its sign would have had the same effect. This is called the "transposition" method It must, however, be remembered that while transposing a term from one side to the other side, the sign of the term must be changed.

In both these examples, our idea is to get the terms involving the variable to the left-hand side and those terms not involving the variable to the right-hand side, and then solve. This can be done by the method of transposition, as explained in the above note. Consider the following examples:

Example 6.7: Solve $2x + 6 = x - 8$.

Solution: Bringing variable terms to the left-hand side and those not involving the variable to right-hand side, we get

$$2x - x = -8 - 6 \text{ (by transposition, changing the signs of 6 and } x\text{)}$$

Thus $x = -14$

So, -14 is the solution.

Example 6.8: Solve $3x + 7 = 20$

Solution: $3x = 20 - 7$ (by transposition)

i.e. $3x = 13$

Hence, $x = \frac{13}{3}$

So, $\frac{13}{3}$ is the solution.

Example 6.9: Solve $2x - 7 = 7x + 4$

Solution: $2x - 7x = 4 + 7$

$\therefore\ -5x = 11$

$\therefore\ x = -\frac{11}{5}$

So, the solution is $-\frac{11}{5}$

Exercises 6.2

1. In the following equations, verify whether the given value of the variable is a solution of the equation:

   (*i*) $x + 4 = 2x ; x = 4.$
   (*ii*) $y - 7 = 3y + 8 ; y = 3.$
   (*iii*) $3u + 2 = 2u + 7 ; u = 5.$
   (*iv*) $2x - 3 = \frac{x}{2} - 2 ; x = \sqrt{2}$
   (*v*) $\frac{5}{2}x + 3 = \frac{21}{2} ; x = 3$
   (*vi*) $24 - 3(u - 2) = u + 8 ; u = -1$
   (*vii*) $(x - 2) + (x + 3) = x + 8 ; x = 0$

2. Solve the following equations:

   (*i*) $3x + 3 = 15$ (*ii*) $2y + 7 = 19$
   (*iii*) $\frac{5}{2}x + 3 = \frac{21}{2}$ (*iv*) $\sqrt{3}x - 2 = 2\sqrt{3} + 4$
   (*v*) $8u + \frac{21}{4} = 3u + 6$ (*vi*) $(\sqrt{5} + 5)x + 4 = 2\sqrt{5} + 8$
   (*vii*) $2x - (3x - 4) = 3x - 5$ (*viii*) $2x + \sqrt{2} = 3x - 4 - 3\sqrt{2}$

6.3 Applications of Linear Equations

In this section we shall use linear equations to solve some simple practical problems. We shall illustrate this by means of the following examples.

Example 6.10: Find the number which, when added to its half, gives 33.

Solution: We denote the number to be found by a variable. So let the number be x.

We then use the given information to formulate a linear equation for x. We do this as follows:

We are given that the number added to its half gives 33.
Hence,

$$x + \frac{1}{2}x = 33$$

This gives us

$$\frac{3}{2}x = 33$$

Solving this we get

$$x = (33) \times \frac{2}{3}$$
$$= 22$$

Thus 22 is the required number.

(Let us verify our solution:
22 + half of 22 = 22 + 11 = 33, as we wanted)

Example 6.11: Ramu is now half his father's age. Twenty years ago the age of Ramu's father was six times Ramu's age. What is Ramu's age now? What is Ramu's father's age now?

Solution: In this example, it looks as though there are two unknown quantities – namely, Ramu's age and Ramu's father's age. But since Ramu is now half his father's age, it is enough to know the age of Ramu's father. Then we can find Ramu's age also. So let x be the present age of Ramu's father in years.

$\therefore$ Ramu's age now $= \frac{1}{2}x$ years

We now use the information given to formulate an equation for x.

Twenty years ago Ramu's father's age $= x - 20$ years

Twenty years ago Ramu's age $= \frac{1}{2}x - 20$ years

$\therefore x - 20 = 6\left(\frac{1}{2}x - 20\right)$, (since we are given that 20 years ago Ramu's father was six times as old as Ramu)

We now solve the linear equation

$$x - 20 = 3x - 120$$
$$\therefore -2x = -100$$
$$\therefore x = 50$$

$\therefore$ The present age of Ramu's father $= x$ years $= 50$ years

Ramu's age now $= \frac{1}{2}x$ years $= 25$ years

(Verify: 20 years ago Ramu was 5 years old, his father was 30 years old, and thus the father was 6 times as old as Ramu).

Example 6.12: The length of a rectangle is 3 cm more than its breadth. If the perimeter of the rectangle is 18 cm, find the length and breadth of the rectangle.

Solution: In this example, again, it appears that there are two unknowns – namely, the length and breadth of the rectangle. However, if we know the length of the rectangle, we can get the breadth by subtracting 3 cm from it. So let x cm be the length of the rectangle.

$$\text{length} = x \text{ cm}$$
$$\therefore \text{breadth} = (x - 3) \text{ cm}$$
$$\therefore \text{Perimeter} = [2x + 2(x - 3)] \text{ cm}$$
$$= (4x - 6) \text{ cm}$$

We are given perimeter = 18 cm

$\therefore 4x - 6 = 18$, giving a linear equation.

Solving, we get

$$4x = 18 + 6 = 24$$
$$\therefore x = 6$$
$$\therefore \text{length} = 6 \text{ cm}; \text{breadth} = 3 \text{ cm}$$

Remarks: Examining the above examples, we find that we follow basically three steps to solve such problems

Step (1): Denote the unknown quantity by x.

Step (2): From the information given, formulate a linear equation for x.

Step (3): Solve the linear equation to find x.

Note: If there are more than one unknown quantities, then in Step 1 call one of these as x and write the others in terms of x

Exercises 6.3

1. A number added to its two-thirds is equal to 35 Find the number.
2. The sum of two numbers, one of which is $\frac{2}{5}$ times the other, is 50 Find the two numbers.
3. In a class with 48 students, the number of girls is $\frac{1}{7}$ times the number of boys. Find the number of boys and the number of girls in the class.
4. A boy is now one-third as old as his father. Twelve years hence he will be half as old as his father. Determine the present age of the boy and that of his father.
5. A man leaves half his property to his wife, one-third to his son and the remaining to his daughter. If the daughter's share is Rs 15,000, how much money did the man leave? How much money did his wife get? What is his son's share?
6. The length of a rectangle is 5 cm more than the breadth. If the perimeter of the rectangle is 40 cm find the length and breadth of the rectangle.
7. The length of a rectangle is 4 cm more than the breadth, and the perimeter is 11 cm more than the breadth. Find the length and breadth of the rectangle
8. The angle A of a triangle ABC is equal to the sum of the other two angles. Also, the ratio of the angle B to the angle C is 4 : 5. Determine the three angles.

9 The sum of two numbers is 50, and their difference is 10. Find the numbers
10. A number consists of two digits The digit at the ten's place is two times the digit at the unit's place. The number, formed by reversing the digits, is 27 less than the original number. Find the original number.
11. 5 years ago, the age of a man was 7 times the age of his son The age of the man will be 3 times the age of his son in five years from now. How old are the man and his son now?
12. The perimeter of a rectangular field is 80 m. If the length of the field is decreased by 2 m and its breadth increased by 2 m, the area is increased by 36 sq.m Find the length and the breadth of the rectangular field
13 Divide Rs 335 among A, B and C, so that A may have Rs 20 more than B, and C Rs 15 more than A.
14 A number consists of two digits of which the ten's digit exceeds the unit's digit by 6. The number itself is equal to ten times the sum of digits. Find the number.
15 Two men start from points A and B, 42 km apart. One walks at 4 km per hour and meets the other, going in the opposite direction, after 6 hours. Find the rate at which the second man is walking.
16 The difference between two numbers is 642. When the greater is divided by the smaller, the quotient is 8 and the remainder is 19. Find the numbers.
17. Divide 300 into two parts so that half of one part may be les[illegible] the other by 48
18 Divide Rs 243 into three parts such that half of the first part, one-third of the second part and one-fourth of the third part, shall be equal
19 By selling a car for Rs 72000, a person made a profit of 20% What was the cost price of the car?
20. On the occasion of Diwali, Khadi Bhandar allows a discount of 20% on all textiles and 25% on ready made garments. Hari paid Rs 180 for a gown. What was the marked price of the gowns?
21. A and B can do a piece of work in 8 days, which A alone can do in 12 days. In how many days can B alone do the same work?
22 A steamer goes downstream from one port to another in 4 hours It covers the same distance upstream in 5 hours. If the speed of the stream be 2 km per hour, find the distance between the two ports.
23. A cycled from P to Q at 10 km per hour and returned at the rate of 9 km per hour. B cycled both ways at 12 km per hour. It was discovered that for the total journey, B took 10 minutes less than A. What is the distance between P and Q?
24. In an election for a Corporation seat, there were two candidates A total of 9791 votes were polled. 116 votes were declared invalid. The successful candidate got 5 votes for every 4 votes his opponent had. By what margin did the successful candidate win?
25. A 700 g dry fruit pack costs Rs 72. It contains some cashew kernel, and the rest as dry grapes. If cashew kernel costs Rs 96 per kg, and dry grapes cost Rs 112 per kg, what were the quantities of the two dry fruits separately?

26. A man invested Rs 35,000; a part of it at an annual interest rate of 12% and the rest at 14% If he received a total annual interest of Rs 4460, how much did he invest at each rate?
27 There are benches in a classroom. If 4 students sit on each bench, three benches are left vacant; and if 3 students sit on each bench, 3 students are left standing. What is the total number of students in the class?
28 If a scooterist drives at the rate of 24 km per hour, he reaches his destination 5 minutes too late; if he drives at the rate of 30 km per hour, he reaches his destination 4 minutes too soon. How far is his destination?
29. Two planes start from a city and fly in opposite directions, one averaging a speed of 40 km/hour greater than that of the other If they are 3400 km apart after 5 hours, find their average speeds.
30. A pharmacist needs to strengthen a 15% alcohol solution to one of 32% alcohol. How much pure alcohol should be added to 400 ml of the 15% solution?

CHAPTER 7

Logarithms

7.1 Rational Powers of a Real Number

You are familiar with powers of rational number and the laws obeyed by their exponents. Now that you are working with real numbers, it is desirable to define powers of real numbers. As you will see in a short while, powers of real numbers are defined much in the same way as powers of rational numbers and the same laws of exponents hold in this case also.

Definition

Integral Power: For any real number a and a positive integer n, we define a^n as

$$a^n = a \times a \ldots \times a \ (n \text{ factors})$$

If m and n are positive integers, and $m > n$, then for $a \neq O$

$$\frac{a^m}{a^n} = \frac{a \times a \times \ldots \times a \ (m \text{ factors})}{a \times a \times . \quad \times a \ (n \text{ factors})}$$

$$= \frac{a \times a \times \ldots \times a[(m-n) \text{ factors}]}{1}$$

$$= a^{m-n}$$

If $m = n$, then $\frac{a^m}{a^m} = a^{m-m} = a^0$, i.e. $a^0 = 1$. Hence, by definition we take $a^0 = 1$.

Again, if we take $m = 0$ in $\frac{a^m}{a^n} = a^{m-n}$, we get $\frac{a^0}{a^n} = \frac{1}{a^n} = a^{0-n} = a^{-n}$

Hence, for a positive integer n, we define $a^{-n} = \frac{1}{a^n}$. We can now say that $\frac{a^m}{a^n} = a^{m-n}$, whether $m > n$, or $m = n$, or $m < n$.

a^n is called the nth power of a. The real number a is called the base and n is called the exponent of the nth power of a.

Illustrations

(*i*) $3^0 = 1$ (*ii*) $2^3 = 2 \times 2 \times 2 = 8$,

(*iii*) $5^{-2} = \frac{1}{5^2} = \frac{1}{25}$

Laws of Integral Exponents

For any two real numbers a and b, $a, b \neq 0$, and for any two positive integers, m and n,

(i) $a^m a^n = a^{m+n}$ (ii) $a^m \div a^n = a^{m-n}$

(iii) $(a^m)^n = a^{mn}$ (iv) $(ab)^n = a^n b^n$

(v) $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

The proofs of these laws follow directly from the definition and we leave it to the student as an exercise.

Illustrations

(i) $5^3 \times 5^4 = 5^{3+4} = 5^7$ (ii) $5^6 \div 5^2 = 5^{6-2} = 5^4$

(iii) $\left(\dfrac{2}{3}\right)^2 \times \left(\dfrac{2}{3}\right)^{-2} = \left(\dfrac{2}{3}\right)^0 = 1$ (iv) $6^4 = (3 \times 2)^4 = 3^4\ 2^4$

(v) $\left(\dfrac{2}{3}\right)^3 = \dfrac{2^3}{3^3} = \dfrac{8}{27}$

We recall here, without proof, the following result.

If a is a positive real number (i e, $a > 0$), and n a natural number, then there exists a unique positive real number x such that $x^n = a$

Definition *(Principal nth root)*

For any real number $a > 0$ and a positive integer n, the principal nth root *of a is the unique positive real number x, such that $x^n = a$.*

The principal nth *root of a positive real number a is denoted by the symbol*

$a^{\frac{1}{n}}$, or $\sqrt[n]{a}$

Remarks: For any real number, $a < 0$, and a positive odd integer n, the principal nth root of a is $-\,|\,a\,|^{\frac{1}{n}}$ For example $(-8)^{\frac{1}{3}} = -\,|-8\,|^{\frac{1}{3}} = -(8^{\frac{1}{3}}) = -2$

For $a < 0$, and n a positive even integer, the principal nth root of a is not defined since an even power of a real number is always positive Therefore $(-16)^{\frac{1}{2}}$ is a meaningless symbol, if we confine ourselves to real numbers only. Later, when you study complex numbers, expressions like $(-16)^{\frac{1}{2}}$ will have a meaning.

Definition *(Rational Exponents)*

For any real number $a > 0$ and a rational number $\dfrac{p}{q}$ $(q > 0)$,

$$a^{\frac{p}{q}} = (a^p)^{\frac{1}{q}}$$

For example: (i) $(4)^{\frac{3}{2}} = (4^3)^{\frac{1}{2}} = (64)^{\frac{1}{2}} = 8$

(ii) $(-4)^{\frac{3}{2}} = [\,(-4)^3\,]^{\frac{1}{2}} = (-64)^{\frac{1}{2}}$, is a meaningless symbol, because there is no real number x such that $x^2 = -64$

We have defined a^m for rational number m only. a^m can also be defined for real values of m but it is beyond the scope of this book to do so. The same laws of exponents as are valid for rational exponents hold for real exponents also.

Laws of Exponents

Based upon definitions of $a^{\frac{1}{q}}$ and $a^{\frac{p}{q}}$, $q > 0$, and the laws of exponents for integral exponents it can be shown that these laws hold for rational exponents as well

Illustrations

(i) $10^{\frac{3}{4}} \cdot 10^{\frac{1}{2}} = 10^{\frac{3}{4}+\frac{1}{2}} = 10^{\frac{5}{4}}$ (ii) $10^{\frac{3}{4}} \div 10^{\frac{1}{2}} = 10^{\frac{3}{4}-\frac{1}{2}} = 10^{\frac{1}{4}}$

(iii) $\left[(16)^{\frac{2}{3}}\right]^{\frac{2}{3}} = 16^{\frac{2}{3}\cdot\frac{2}{3}} = 16^{\frac{4}{9}}$

(iv) $(50)^{\frac{1}{2}} = (25 \times 2)^{\frac{1}{2}} = (25)^{\frac{1}{2}} \cdot 2^{\frac{1}{2}} = 5\,2^{\frac{1}{2}}$

(v) $\left(\frac{3}{5}\right)^{\frac{1}{3}} = \dfrac{3^{\frac{1}{3}}}{5^{\frac{1}{3}}}$

We illustrate the applications of these laws by means of a few more examples In many cases it is convenient to work with radicals rather than with rational exponents. Although π is not a rational number, yet we may use $\sqrt{\pi}$ instead of $\pi^{\frac{1}{2}}$

Example 7.1: Simplify: (i) $\pi^{\frac{3}{4}} \cdot \pi^{\frac{1}{2}}$, (ii) $\pi^{\frac{3}{4}} \div \pi^{\frac{1}{2}}$

Solution: (i) $\pi^{\frac{3}{4}} \cdot \pi^{\frac{1}{2}} = \pi^{\frac{3}{4}+\frac{1}{2}} = \pi^{\frac{5}{4}}$,

(ii) $\pi^{\frac{3}{4}} \div \pi^{\frac{1}{2}} = \pi^{\frac{3}{4}-\frac{1}{2}} = \pi^{\frac{1}{4}}$

Example 7.2: $(\sqrt{2}\,\pi)^{\frac{1}{2}} = (\sqrt{2})^{\frac{1}{2}}\ \pi^{\frac{1}{2}} = 2^{\frac{1}{4}}\ \pi^{\frac{1}{2}}$

Example 7.3: If x be a non-zero real number, and l, m, n positive integers, show that

$$\left(\frac{x^m}{x^n}\right)^l \left(\frac{x^n}{x^l}\right)^m \left(\frac{x^l}{x^m}\right)^n = 1$$

Solution: The expression

$$\begin{aligned}\left(\frac{x^m}{x^n}\right)^l \left(\frac{x^n}{x^l}\right)^m \left(\frac{x^l}{x^m}\right)^n &= (x^{m-n})^l\ (x^{n-l})^m \cdot (x^{l-m})^n \\ &= x^{lm-ln} \cdot x^{nm-lm} \cdot x^{ln-mn} \\ &= x^{lm-ln+mn-lm+ln-mn} \\ &= x^0 \\ &= 1\end{aligned}$$

Example 7.4: Simplify: $(12)^{-1/2} \cdot \sqrt[3]{25}$

Solution: $(12)^{-\frac{1}{2}} \times \sqrt[3]{25} = (4 \times 3)^{-\frac{1}{2}} \times \sqrt[3]{5^2}$

$$= 4^{-\frac{1}{2}} . 3^{-\frac{1}{2}} . 5^{\frac{2}{3}}$$

$$= \frac{1}{4^{\frac{1}{2}}} . \frac{1}{3^{\frac{1}{2}}} . 5^{\frac{2}{3}}$$

$$= \frac{1}{2} . \frac{3^{\frac{1}{2}}}{3^1} \quad 5^{\frac{2}{3}}$$

$$= \frac{1}{6} . 3^{\frac{1}{2}} \quad 5^{\frac{2}{3}}$$

Exercises 7.1

1 Simplify each of the following, removing radical signs and negative indices wherever they occur:

(i) $(\sqrt{4})^{-\frac{3}{4}}$ (ii) $(\sqrt{5})^{-3}.(\sqrt{2})^{-3}$

(iii) $\dfrac{1}{\sqrt[3]{4^{-5}}}$ (iv) $(25)^{-\frac{1}{3}} \times \sqrt[3]{16}$

(v) $(\sqrt[3]{8})^{-\frac{1}{2}}$ (vi) $(\sqrt{4})^{-7}(\sqrt{2})^{-5}$

2 Assuming that x, y, z are positive real numbers, simplify each of the following:

(i) $\sqrt{x^{-2} y^3}$ (ii) $(x^{-\frac{2}{3}} . y^{-\frac{1}{2}})^2$

(iii) $(\sqrt{x}^{-3})^5$ (iv) $(\sqrt{x})^{-\frac{2}{3}} \sqrt{y^4} \div \sqrt{xy^{-\frac{1}{2}}}$

(v) $\sqrt[3]{xy^2} \div \sqrt{x^2 y}$. (vi) $\sqrt[4]{\sqrt[3]{x^2}}$

3 Assuming that x, y, z are positive real numbers and the exponents are all rational numbers, show that

(i) $\left(\dfrac{x^a}{x^b}\right)^{a^2+ab+b^2} . \left(\dfrac{x^b}{x^c}\right)^{b^2+bc+c^2} . \left(\dfrac{x^c}{x^a}\right)^{c^2+ca+a^2} = 1$

(ii) $\sqrt{x^{-1} y} . \sqrt{y^{-1} z} . \sqrt{z^{-1} x} = 1$

(iii) $\left(\dfrac{x^a}{x^b}\right)^{\frac{1}{ab}} \left(\dfrac{x^b}{x^c}\right)^{\frac{1}{bc}} \left(\dfrac{x^c}{x^a}\right)^{\frac{1}{ca}} = 1$

(iv) $\dfrac{1}{1 + x^{b-a} + x^{c-a}} + \dfrac{1}{1 + x^{a-b} + x^{c-b}} + \dfrac{1}{1 + x^{b-c} + x^{a-c}} = 1$

7.2 Definition and Laws of Logarithms

Sometimes, we have to work with large numbers A numerical expression may involve multiplication, division or rational powers of large numbers For such calculations, logarithms are very useful They help us in making difficult calculations easy We shall first introduce this new concept, and discuss the laws, which will have to be followed in working with logarithms, and then apply this technique to a number of problems to show how it makes difficult calculations simple.

We know that

$$2^3 = 8, 3^2 = 9, 5^3 = 125, 7^0 = 1$$

In general, for a positive real number a, and a rational number m, let

$$a^m = b,$$

where b is a real number In other words

the mth power of base a is b.

Another way of stating the same fact is

logarithm of b to base a is m.

Definition

If for a positive real number a, $a \neq 1$

$$a^m = b,$$

we say that m is the logarithm of b to the base a.

We write this as

$$\log_a b = m$$

"log" being the abbreviation of the word "logarithm."

Thus, we have

$\log_2 8 = 3,$ since $2^3 = 8$

$\log_3 9 = 2,$ since $3^2 = 9$

$\log_3 9 = 3,$ since $5^3 = 125$

$\log_7 1 = 0,$ since $7^0 = 1$

Some other examples are

Since $\sqrt{9} = 3$ or $9^{\frac{1}{2}} = 3,$ $\log_9 3 = \frac{1}{2}$

Since $16^{\frac{1}{4}} = 2,$ $\log_{16} 2 = \frac{1}{4}$

Exercises 7.2

1 Write the following in the form of logrithms:

(i) $2^5 = 32$ (ii) $10^3 = 1000$ (iii) $3^4 = 81$

(iv) $5^4 = 625$ (v) $10^{-1} = 0.1$ (vi) $7^2 = 49$

2. Express each of the following in exponential form.

(i) $\log_5 25 = 2$ (ii) $\log_3 243 = 5$ (iii) $\log_{10} 1000 = 3$

(iv) $\log_2 64 = 6$ (v) $\log_4 64 = 3$ (vi) $\log_{10} \cdot 01 = -2$

Laws of Logarithms

In the following discussion, we shall take logarithms to any base a, ($a > 0$ and $a \neq 1$).

First Law: $\log_a (mn) = \log_a m + \log_a n$

Proof: Suppose that $\log_a m = x$ and $\log_a n = y$

Then $a^x = m, a^y = n$

Hence $mn = a^x \; a^y = a^{x+y}$

It now follows from the definition of logarithms that

$\log_a (mn) = x + y = \log_a m + \log_a n$

Second Law: $\log_a(\frac{m}{n}) = \log_a m - \log_a n$

Proof: Let $\log_a m = x$, $\log_a n = y$

Then $a^x = m, a^y = n$

Hence $\frac{m}{n} = \frac{a^x}{a^y} = a^{x-y}$

Therefore

$\log_a (\frac{m}{n}) = x - y = \log_a m - \log_a n$

Third Law:

$\log_a (m^n) = n \log_a m$

Proof: As before, if $\log_a m = x$, then $a^x = m$

Then $m^n = (a^x)^n = a^{nx}$

giving $\log_a (m^n) = nx = n \log_a m$

Remarks: In words, the First Law says: *The log of the product of two numbers is equal to the sum of their logs*. Similarly, the Second Law says: *the log of the ratio of two numbers is the difference of their logs*. Thus, the use of these laws converts a problem of multiplication / division into a problem of addition / subtraction, which are far easier to perform than multiplication / division. That is why logarithms are so useful in all numerical computations.

Exercises 7.3

In each of the following, assume that the base $a = 10$ wherever it has not been indicated:

1. Prove that $\log (mnp) = \log m + \log n + \log p$
2. Prove that $\log (a_1 . a_2 \, .. a_k) = \log a_1 + \log a_2 ... + \log a_k$.
3. Prove that $\log 12 = \log 3 + \log 4$.

4 Show that $\log 360 = 3\log 2 + 2\log 3 + \log 5$

5. Show that $\log\frac{50}{147} = \log 2 + 2\log 5 - \log 3 - 2\log 7$

6. Show that

(i) $3\log 2 + \log 5 = \log 40$

(ii) $5\log 3 - \log 9 = \log 27$

7.3 Logarithms to Base 10

Because the number 10 is the base of writing numbers, it is very convenient to use logarithms to the base 10. Some examples are.

$\log_{10} 10 = 1$, since $10^1 = 10$

$\log_{10} 100 = 2$, since $10^2 = 100$

$\log_{10} 10{,}000 = 4$, since $10^4 = 10{,}000$

$\log_{10} 0.01 = -2$, since $10^{-2} = .01$

$\log_{10} 0.001 = -3$, since $10^{-3} = .001$,

and $\log_{10} 1 = 0$, , since $10^0 = 1$

The above results indicate that if n is an integral power of 10, i.e., 1 followed by several zeros or 1 preceded by several zeros immediately to the right of the decimal point, then $\log n$ can be easily found

If n is not an integral power of 10, then it is not easy to calculate $\log n$. But mathematicians have made tables from which we can read off approximate value of the logarithm of any positive number between 1 and 10 And these are sufficient for us to calculate the logarithm of any number expressed in decimal form. For this purpose, we always express the given decimal as the product of an integral power of 10 and a number between 1 and 10.

Standard Form of Decimal

We can express any number in decimal form, as the product of (i) an integral power of 10, and (ii) a number between 1 and 10. Here are some examples:

(i) 25.2 lies between 10 and 100

$$25.2 = \frac{25.2}{10} \times 10 = 2.52 \times 10^1.$$

(ii) 1038.4 lies between 1000 and 10000

$$\therefore 1038.4 = \frac{1038.4}{1000} \times 10^3 = 1.0384 \times 10^3$$

(iii) .005 lies between .001 and .01.

$$\therefore .005 = (.005 \times 1000) \times 10^{-3} = 5.0 \times 10^{-3}$$

(iv) 0.00025 lies between .0001 and .001.

$$\therefore .00025 = (.00025 \times 10000) \times 10^{-4} = 2.5 \times 10^{-4}$$

In each case, we divide or multiply the decimal by a power of 10, to bring one non-zero digit to the left of the decimal point, and do the reverse operation by the same power of 10, indicated separately.

Thus, any positive decimal can be written in the form

$$n = m \times 10^p$$

where p is an integer (positive, zero or negative) and $1 \leq m < 10$. This is called the "*standard form of n.*"

Working Rule

(1) Move the decimal point to the left, or to the right, as may be necessary, to bring one non-zero digit to the left of decimal point.

(2) (i) If you move p places to the left, multiply by 10^p

(ii) if you move p places to the right, multiply by 10^{-p}

(iii) If you do not move the decimal point at all, multiply by 10^0

(iv) Write the new decimal obtained by the power of 10 (of Step 2) to obtain the standard form of the given decimal.

Exercises 7.4

1 Write each of the following in standard form·

(*i*) 3.123, (*ii*) 31.23 (*iii*) 312.3

(*iv*) 3123 (*v*) 312300 (*vi*) 0.3123

(*vii*) .03123

2. Write the following numbers in decimal form, without powers of 10 as factors.

(*i*) 5.6×10^3 (*ii*) 1.436×10^{-1}

(*iii*) 2.4×10^{-2} (*iv*) 9.632×10^5

(*v*) 1.2056×10^2 (*vi*) $1\ 2056 \times 10^{-2}$

7.4 Characteristic and Mantissa

Consider the standard form of n

$$n = m \times 10^p, \text{ where } 1 \leq m < 10$$

Taking logarithms to the base 10 and using the laws of logarithms

$$\begin{aligned} \log n &= \log m + \log 10^p \\ &= \log m + p \log 10 \\ &= p + \log m \end{aligned}$$

Here p is an integer and as $1 \leq m < 10$, so $0 \leq \log m < 1$, i e., $\log m$ lies between 0 and 1 When $\log n$ has been expressed as $p + \log m$, where p is an integer and $0 \leq \log m < 1$, we say that p is the "*characteristic*" of $\log n$ and that $\log m$ is the "*mantissa*" of $\log n$ Note that characteristic is always an integer – positive, negative or zero, and mantissa is never negative and is always less than 1 If we can find the characteristic and the mantissa of $\log n$, we have to just add them to get $\log n$.

Thus, to find $\log n$, all we have to do is as follows.

1. Put n in the standard form, say

$$n = m \times 10^p,\ 1 \leq m < 10$$

2. *Read off the characteristic p of log n from this expression (exponent of 10).*
3 *Look up log m from tables, which is being explained below*
4. *Write log n = p + log m*

If the characteristic p of a number n is say, 2 and the mantissa is .4133, then we have logn = 2 + .4133 which we can write as 2 4133 If, however, the characteristic p of a number m is say -2 and the mantissa is .4123, then we have logm = $-2 + .4123$ We cannot write this as -2.4123. (Why?) In order to avoid this confusion we write $\bar{2}$ for -2 and thus we write logm = $\bar{2}$ 4123.

Now let us explain how to use the table of logarithms to find mantissas. A table is appended at the end of this book.

Observe that in the table, every row starts with a two digit number, 10, 11, 12, . . . 97, 98, 99 Every column is headed by a one-digit number, 0, 1, 2, . . . 9 On the right, we have the section called "Mean differences" which has 9 columns headed by 1, 2 . 9

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
61	7853	7860	7868	7875	7882	7889	7896	7803	7810	7817	1	1	2	3	4	4	5	6	6
62	7924	7931	7935	$\boxed{7945}$	7954	7959	7966	7973	7980	7987	1	1	2	$\boxed{3}$	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	6	6

Now suppose we wish to find log (6.234) Then look into the row starting with 62 In this row, look at the number in the column headed by 3. The number is 7945. This means that

$$\log (6.230) = 0.7945^*$$

But we want log (6.234). So our answer will be a little more than 0 7945. How much more? We look this up in the section on Mean differences. Since our fourth digit is 4, look under the column headed by 4 in the Mean difference section (in the row 62). We see the number 3 there. So add 3 to 7945. We get 7948 So we finally have

$$\log (6.234) = 0.7948$$

Take another example. To find log (8.127), we look in the row 81 under column 2, and we find 9096 We continue in the same row and see that the mean difference under 7 is 4 Adding this to 9096, and we get 9100. So, log (8 127) = 0 9100

*It should, however, be noted that the values given in the table are not exact They are only approximate values, although we use the sign of equality which may give the impression that they are exact values The same convention will be followed in respect of antilogarithm of a number

Exercises 7.5

1. Use logarithm tables to find the logarithms of the following numbers

(*i*) 1270	(*ii*) 12.70	(*iii*) 431 5
(*iv*) 11 23	(*v*) 0.1257	(*vi*) 0.0012
(*vii*) 0 00001379		

7.5 Finding N when log N is Given

We have so far discussed the procedure for finding $\log n$ when a positive number n is given. We now turn to its converse i e , to find n when $\log n$ is given and give a method for this purpose If $\log n = t$, we sometimes say n = *antilog t* Therefore our task is given t, find its antilog. For this, we use the ready-made antilog tables (One is included at the end of this book).

Suppose $\log n = 2\,5372$

To find n, first take just the mantissa of $\log n$. In this case it is .5372 (Make sure that it is positive) Now take up antilog of this number in the antilog table which is to be used exactly like the log table. In the antilog table, the entry under column 7 in the row 53 is 3443 and the mean difference for the last digit 2 in that row is 2, so the table gives 3445 Hence,

antilog (5372) = 3 445

Now since $\log n = 2.5372$, the characteristic of $\log n$ is 2. So the standard form of n is given by

$$n = 3\,445 \times 10^2$$

$$\text{or } n = 344.5$$

Let us calculate some more antilogs.

Example 1: If $\log x = 1\,0712$, find x

Solution We find that the number corresponding to 0712 is 1179. Since characteristic of $\log x$ is 1, we have

$$x = 1\,179 \times 10^1$$
$$= 11\,79$$

Example 2: If $\log x = \bar{2}\,1352$, find x

Solution·From antilog tables, we find that the number corresponding to 1352 is 1366 Since the characteristic is $\bar{2}$, i.e., -2, so

$$x = 1.366 \times 10^{-2} = 0.01366$$

Exercises 7.6

1 Using tables, find the logarithm of each of the following numbers·

(*i*) 4 8	(*ii*) 7	(*iii*) 3 17
(*iv*) 3.172	(*v*) .235	(*vi*) 2354

2 Find $\log x$, if x equals

(i) .0768 (ii) .0025 (iii) .0087
(iv) .00954 (v) .0056 (vi) .0287

3 Find the antilogarithm of each of the following

(i) 0.752 (ii) $.\underline{3}01$ (iii) $.\underline{5}428$
(iv) 2.752 (v) $\bar{1}.301$ (vi) $\bar{2}.5428$

4 Each of the following numbers is the logarithm of some number. Express each in the form $p + \log m$, where p is the characteristic and $\log m$ the mantissa, and find the number.

(i) 1.2086 (ii) −1.2084 (iii) −2.4325
(iv) −3.6432 (v) 2.5674 (vi) −0.62

7.6 Use of Logarithms in Numerical Calculations

Example 7.5: Find 6.3×1.9.

Solution: Let $x = 6.3 \times 1.9$

Then $\log x = \log(6.3 \times 1.9) = \log 6.3 + \log 1.9$

Now,

$$\log 6.3 = 0.7993$$
$$+ \log 1.9 = 0.1106$$
$$\therefore \log x = 0.9099,$$

$\therefore$ Taking antilog

$$x = 8.127$$

Example 7.6: Find $\dfrac{(1.23)^{1.5}}{11.2 \times 23.6}$

Solution: Let $x = \dfrac{(1.23)^{\frac{3}{2}}}{11.2 \times 23.6}$

Then $\log x = \log \dfrac{(1.23)^{\frac{3}{2}}}{11.2 \times 23.6}$

$$= \frac{3}{2} \log 1.23 - \log(11.2 \times 23.6)$$
$$= \frac{3}{2} \log 1.23 - \log 11.2 - \log 23.6$$

Now,

$$\log 1.23 = 0.0899$$
$$\frac{3}{2} \log 1.23 = 0.13485$$
$$\log 11.2 = 1.0492$$
$$\log 23.6 = 1.3711$$
$$\log x = \underline{0}.13485 - 1.0492 - 1.3711$$
$$= \bar{3}.71455$$
$$\therefore x = 0.005183$$

Example 7.7: Find $\sqrt{\dfrac{(71\ 24)^5 \times \sqrt{56}}{(2\ 3)^7 \times \sqrt{21}}}$

Solution: Let $x = \sqrt{\dfrac{(71\ 24)^5 \times \sqrt{56}}{(2\ 3)^7 \times \sqrt{21}}}$

Then $\log x = \frac{1}{2} \log \left[\dfrac{(71.24)^5 \times \sqrt{56}}{(2.3)^7 \times \sqrt{21}}\right]$

$= \frac{1}{2} [\log (71\ 24)^5 + \log \sqrt{56} - \log (2\ 3)^7 - \log \sqrt{21}]$

$= \frac{5}{2} \log 71.24 + \frac{1}{4} \log 56 - \frac{7}{2} \log 2.3 - \frac{1}{4} \log 21$

Now, using log tables

$\log 71\ 24 = 1.8527$

$\log 56 = 1.7482$

$\log 2\ 3 = 0.3167$

$\log 21 = 1\ 3222$

$\therefore \log x = \frac{5}{2} (1.8527) + \frac{1}{4} (1.7482) - \frac{7}{2} (0\ 3617) - \frac{1}{4} (1\ 3222)$

$= 3\ 4723$

$\therefore \quad x = 2967$

Example 7.8: Find the approximate area of a circular plot of land, whose radius is 75 m (Take $\log \pi = 0.4972$.)

Solution: If A be the area in sq metres, then we know

$A = \pi r^2 = \pi(75)^2$

$\log A = \log \pi(75)^2 = \log \pi + 2 \log 75$

Now, $\log 75 = 1.8751$

$\log \pi = 0\ 4972$

$\therefore \log A = 0\ 4972 + 2\ (1\ 8751)$

$= 4.2474$

$\therefore A = 17680$

Hence, area required $= 17680\ m^2$

Exercises 7.7

(Use logarithmic tables.)

1. Find the value of

   (i) 6.45×981.4 (ii) 0.0064×1.507

2. Evaluate:

   (i) $\dfrac{2.632}{\cdot 0045}$ (ii) $054 - 216.3$

3. Find the value of

   (i) $(724)^3$ (ii) $\sqrt{42\ 36}$

4 Find the cube root of 48, correct to two decimal places

5. Write down the logarithm of 2^{64} and use it to state the number of digits in the numeral for 2^{64}

6. Using logarithmic tables, evaluate:

   (i) $\frac{8\ 25 \times 4.63}{2.18}$ (ii) $1045 \times \frac{27}{49}$

   (iii) $\frac{22}{7} \times (3.2)^2$ (iv) $\frac{4}{3} \times 3.142 \times (1.5)3$

7 Find the values of the following

   (a) $\sqrt[3]{0.0847}$ (b) $(0\ 09634)^3$

   (c) $\sqrt[4]{.6789}$ (d) $\sqrt[5]{42\ 7}$

8 Using logarithms, Simplify

   (a) 57.12×2.034 (b) $0.8623 \times 0\ 000451$

   (c) $352.6 \times 0\ 078\ 0\ 5943$ (d) $2456 \times 0\ 000071$

   (e) $328\ 4 \times 12.65$ (f) $0.3865 - 0\ 000572$

   (g) $0.0953 \div 3\ 794$ (h) $25 \div 0\ 0683$

9. Find the values of the following:

   (a) $\frac{(25.36)^2 \times 0.4569}{847\ 5}$ (b) $\sqrt{\frac{41.32 \times 20\ 18}{12.69}}$

10 Evaluate, correct to three decimal places·

   (a) $\frac{563\ 4 \times \sqrt[3]{0\ 4573}}{(6.15)^3}$ (b) $\frac{(73\ 56)^3 \times (0\ 0371)^2}{68.21}$

   (c) $\sqrt[3]{\frac{31\ 42 \times 7.192}{(0.236)^2}}$ (d) $\sqrt[3]{\frac{(45\ 4)^2}{(3\ 2)^2 \times (5.6)^3}}$

7.7 Applications

Compound Interest:

When the interest is added to the principal at regular specified intervals of time, so that the amount at the end of an interval becomes the principal for the next interval, then total interest over all the intervals, calculated in this way is called "compound interest".

In any problem, it is more convenient to find the total amount and obtain the compound interest by subtracting the principal from it.

Now, if P is the principal, n the number of intervals, r is the rate of interest percent per interval, and A the total amount at the end of the intervals, then you may recall that

$$A = P\left(1 + \frac{r}{100}\right)^n$$

Given any three of the variables P, r, n and A, you can easily determine the fourth. Ofcourse, to avoid lengthy calculations, the logarithms are available to us now

Example: 7.9: If Rs 273 are invested at 12 % interest, compounded every year, for 5 years, what is the amount realised at the end of 5 years?

Solution: We know the compound interest formula

$$A = P\left(1 + \frac{r}{100}\right)^n,$$

where P is the principal sum, r the rate percent of interest, n the number of years, and A the total amount at the end.

Here, P = Rs 273, $r = 12$, $n = 5$ and A is to be determined.

If A = Rs x, then

$x = 273\,(1 + .12)^5$

or $x = 273\,(1.12)^5$

$\therefore \log x = \log 273 + 5 \log 1.12$

We have

$\log 273 = 2.4362$

$5 \log 1.12 = .2460$

$\therefore \log x = 2.6822$

Hence, $x = 481.0$

$\therefore$ Required Amount = Rs 481.00 (approx.)

Example 7.10 A Colour TV, can be purchased outright for Rs 12500 or in the instalment plan by paying Rs 4500, followed by 11 monthly instalments each of Rs 850. What is the difference between the cash price and the instalment price? What rate of interest does the buyer pay?

Solution: It is clear that if you buy on instalment basis you pay in all Rs 4500 + Rs (11 × 850) = Rs (4500 + 9350) = Rs 13850. So the difference between the cash price (Rs 12500) and the instalment price (Rs 13850) is Rs 1350, which is the interest paid by the buyer. This can be thought of as the total interest charged on 11 loans, each of Rs 850, for varying periods of time.

The 1st loan of Rs 850 for 1 month;

The 2nd loan of Rs 850 for 2 months and so on;

Thus, the 11th loan of Rs 850 for 11 months.

This is equivalent to one loan of Rs 850 for (1 + 2 + 3 + 4 + . . . + 11) = 66 months. If the rate of interest is r%, then the total interest on Rs 850 for 66 months or $\frac{66}{12}$ years is

$$= 850 \times \frac{r}{100} \times \frac{66}{12}$$

$$\therefore 1350 = \frac{850 \times 66 \times r}{1200}$$

$$\text{or } r = \frac{1350 \times 1200}{850 \times 66}$$

$$\therefore \log r = \log 1350 + \log 1200 - \log 850 - \log 66$$

We have

$$\log 1350 = 3.1303$$
$$\log 1200 = 3.0792$$
$$\log 85 = 2.9294$$
$$\log 66 = 1.8195$$
$$\therefore \log x = 1.4606$$
$$\text{and } x = 28.90$$

$\therefore$ Rate of interest paid by the buyer

$$= 29\% \text{ (approx.)}$$

Remarks: The rate of interest calculated here is on the understanding that simple interest was charged

Population Growth

When an entity increases in magnitude over a period of time, we say that it has grown during that period. The growth is measured as a ratio of the increase in magnitude of the entity to its initial magnitude. In other words, growth is relative increase in magnitude

Often, we observe the magnitude of a growing entity at regular intervals. If P_0 is the magnitude in the beginning of a unit of time and P_1 the magnitude at the end of the unit of time, then the ratio

$$\frac{P_1 - P_0}{P_0}$$

is the growth in one unit of time. We may call it the "rate of growth." Thus,

rate of growth = growth per unit time.

As usual we shall express rate of growth as a percentage Thus, if

$$\frac{P_1 - P_0}{P_0} = \frac{r}{100} \quad \text{or } P_1 = P_0 (1 + \frac{r}{100})$$

then we say the rate of growth is r % per unit of time

Let us suppose P_0 is the population at the beginning of a certain year and $r\%$ is the constant rate of growth per year. Then population after one year is

$$P_1 = p_0 (1 + \frac{r}{100})$$

After two years, we have

$$P_2 = P_1 (1 + \frac{r}{100}) = P_0 (1 + \frac{r}{100})^2$$

$$P_3 = P_0 (1 + \frac{r}{100})^3 \text{ and so on.}$$

For a positive integer n, population after n years will be

$$P_n = P_0 (1 + \frac{r}{100})^n$$

This is just like the formula for compound amount, all calculations are done the same way

Example 7.11: In the 1981 census, the population of India was found to be 6 7 $\times 10^7$ If the population increases at the rate of 2.5 % every year, what would be the population in 1991?

Solution: This is a case of compound growth at the rate of 2 5%. So the formula

$$x = P\left(1 + \frac{r}{100}\right)^n$$

is applicable.

Here $P = 6 7 \times 10^7$, $r = 2 5$, $n = 10$, and x is population at the end of 10 years.

$$\therefore\ x = 6 7 \times 10^7 \times \left(1 + \frac{2 5}{100}\right)^{10}$$
$$= 6.7 \times 10^7 \times (1.025)^{10}$$

Taking logs, on either side

$$\log x = \log\left[(6.7 \times 10^7) \times (1 025)^{10}\right]$$

$$= \log (6 7 \times 10^7) + 10 \log 1 025$$

We have,

$$\log(6.7 \times 10^7) = 7.8261$$
$$10 \log 1.025 = 0 1070$$
$$\therefore\ \log x = 7.9331$$

Hence, $x = 8 572 \times 10^7$

$\therefore$ Required population = 8.572×10^7

Example 7.12

In 1980, the population of the state of Andhra Pradesh and of France was roughly the same (about 5.4×10^7). If the population of Andhra Pradesh grows at the rate of 2.4% per year and that of France grows at the rate of 1.8% per year, show that in the year 2100, the population of Andhra Pradesh will be roughly twice that of France

Solution: Population of Andhra Pradesh in 2100 (i.e., 120 years after 1980) will be

$$P_A = P\left(1 + \frac{2.4}{100}\right)^{120}$$

and that of France in 2100 will be

$$P_F = P\left(1 + \frac{1.8}{100}\right)^{120}$$

where P is the present population of the two countries.

$$\text{Now } \log\left(1 + \frac{2\,4}{100}\right)^{120} = 120 \log (1.024)$$
$$= 120 \times 0.0103$$
$$= 1.236$$
$$\text{And } \log\left(1 + \frac{1.8}{100}\right)^{120} = 120 \log (1.018)$$
$$= 120 \times 0.0078$$
$$= 0\ 936$$
$$\therefore \log\frac{P_A}{P_F} = \log (P_A) - \log (P_F)$$
$$= 1\ 236 - 0\ 936$$
$$= 0.300$$
$$\therefore \frac{P_A}{P_F} = \text{antilog } 0\ 300$$
$$= 1\ 995$$
$$= 2 \text{ approx.}$$

In the year 2100, population of Andhra Pradesh will be roughly twice that of France

Depreciation of Value

The value of a machine or of any other article subject to wear and tear decreases with time Relative decrease in the value of a machine is called its "depreciation." In other words, depreciation (or decay) is negative growth.

Rate of depreciation is depreciation per unit time

Thus, if V is the value as a certain time, and $r\%$ is the rate of depreciation per year, the value V_t at the end of t years is

$$V_t = V_0 \left(1 - \frac{r}{100}\right)^t$$

Example 7.13: A machine purchased for Rs 10,000 depreciates at the rate of 6% per annum, the depreciation being worked out on the value of the machine at the beginning of the year. Use log tables to obtain its depreciated value after 7 years (Given log 6 = .77815, log 9 4 = 97313, log 64 85 = 1.81192)

Solution: Since the rate of depreciation is 6%, if the value of the machine is Re 1 at the beginning of the year, its depreciated value is Rs $(1 - \frac{6}{100})$ at the end of the year

depreciated value of Re 1 at the end of 7 years = Rs $(1 - \frac{6}{100})^7$

The purchase price of the machine = Rs 10,000

Suppose depreciated value at the end of 7 years is Rs x Then

$$x = 10{,}000 \times (1-.06)^7$$
$$\log x = \log 10{,}000 + 7\log(.94)$$
$$= 4 + 7 \times (\bar{1}.9731)$$
$$= 4 + \bar{1}.8117$$
$$= 3.8117$$
$$\therefore x = 6482$$

Hence, depreciated value required

$$= \text{Rs } 6482 \text{ (approx.)}$$

Mensuration

You may recall the following formulae for the areas of some plane figures

(i) Area of Rectangle (sides: a cm, b cm);

$$A = ab \text{ cm}^2$$

(ii) Area of a Parallelogram (base a cm, height h cm)

$$A = ah \text{ cm}^2$$

(iii) Area of a Triangle (base: a cm, height . h cm)

$$A = \frac{1}{2} ah \text{ cm}^2$$

(iv) Area of a Triangle [sides $a, b, c, 2s = a + b + c$]

$$A = \sqrt{s(s-a)(s-b)(s-c)} \text{ cm}^2$$

(Hero's Formula)

(v) Area of a Rhombus (diagonals. d_1 cm, d_2 cm)

$$A = \frac{1}{2}(d_1 \times d_2)\text{cm}^2$$

(vi) Area of a Trapezium (parallel sides a cm, b cm, height h cm)

$$A = \frac{1}{2} h (a+b) \text{ cm}^2$$

Example 7.14: The lengths of the parallel sides of a trapezium are 56 cm and 40 cm, and the lengths of the other sides are 28 cm and 30 cm Find the area of the trapezium

Solution: Let $ABCD$ be the trapezium, such that $AB \parallel CD$, and

$AB = 56$ cm, $CD = 40$ cm

$AD = 30$ cm, $BC = 28$ cm

Let $CE \parallel AD$, meet AB in E, so that $AECD$ is a parallelogram As such

$CE = 30$ cm

Now, in ΔEBC,

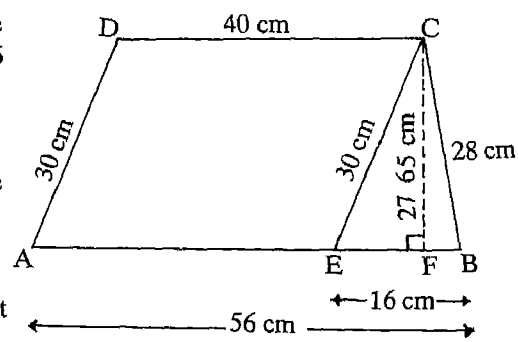

Fig 7.1

$$\text{perimeter } 2s = (30 + 28 + 16)\text{cm}$$
$$= 74 \text{ cm}$$

$$\therefore \text{ Area of } \Delta EBC = \sqrt{s(s-a)(s-b)(s-c)} \text{ cm}^2$$
$$= \sqrt{37(37-16)(37-28)(37-30)} \text{ cm}^2$$
$$= \sqrt{37 \times 21 \times 9 \times 7} \text{ cm}^2$$

Let this area be x cm^2, then

$$x = \sqrt{37 \times 21 \times 9 \times 7}$$
$$\log x = \frac{1}{2}(\log 37 + \log 21 + \log 9 + \log 7)$$
$$= \frac{1}{2}(1.5682 + 1.3222 + 0\ 9542 + 0.8451)$$
$$= \frac{1}{2}(4\ 6897)$$
$$= 2.3449$$
$$\therefore x = 221\ 2$$

Hence, area of Δ EBC $= 221\ 2$ cm^2

Suppose, p cm is the altitude CF from C_1 then

$$p = 2 \times \frac{221.2}{16}$$
$$= 27.65$$
$$\therefore CF = 27\ 65 \text{ cm}$$

Now the area of the trapezium $ABCD$

$$= \frac{1}{2} \times (\text{height}) \times (\text{sum of sides})$$
$$= \frac{1}{2} \times 27\ 65 \times (56 + 40) \text{ cm}^2$$
$$= 48 \times 27.65 \text{ cm}^2$$
$$= x \text{ cm}^2 \text{ (say)}$$

then $x = 48 \times 27\ 65$

$$\log x = \log 48 + \log 27\ 65$$
$$= 1\ 6812 + 1.4417$$
$$= 3.1229$$

and $x = 1327$

Hence, area of trapezium $= 1327$ cm^2

Exercises 7.8

1. A savings bank account pays 5% interest (per year) and it is compounded every six months When a boy is 13 years old, Rs 100 is deposited to his credit in a savings bank account. How much is due to him when he is 21 years old?
2. In how many years will an account double itself at 5% interest compounded annually?
3. A nationalised bank issues "Re-investment Certificates" for a period of 3 years. If Rs 5000 is invested in these certificates, their maturity value is Rs 6725. Assuming that the interest is compounded ever year, what is the rate of interest?
4. The cash price of a new car is Rs 90,000. The insurance company calculates its price at any subsequent time according to the rule that the price depreciates at the rate of 5% a year during the first two years and at the rate of 10% a year thereafter. What will be the price of the car after (a) 2 years, (b) 5 years, (c) 10 years? When will the price be half the original price?
5. The cash price of a refrigerator is Rs 5500 It can also be purchased for Rs 2000 followed by 11 equal monthly instalments of Rs 400 each What is the rate of interest charged under the instalment plan?
6. The population of Pakistan is 8.25×10^7, and that of the State of Uttar Pradesh in India is 1.11×10^8. If the annual rate of growth in Pakistan and U.P. are 2.7% and 2.6% respectively, when will Pakistan's population catch up with that of U.P.?
7. The area of a rectangular field is 2.5 hectares and its sides are in the ratio 3:2. Find the perimeter of the field.
8. The sides of a triangular plate are 8 cm, 19 cm and 15 cm. If its weight be 96 gram, what is the weight of the plate per square cm?
9. Each side of an equilateral triangle is 32.4 cm. Find its area.
10. The lengths of adjacent sides of a parallelogram are 51 cm and 37 m. The length of one of the diagonals is 20 m. Find its smaller altitude.

N	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	5 9 13 4 8 12	17 21 26 16 20 24	30 34 38 28 32 36
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4 8 12 4 7 11	16 20 23 15 18 22	27 31 35 26 29 33
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3 7 11 3 7 10	14 18 21 14 17 20	25 28 32 24 27 31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3 6 10 3 7 10	13 16 19 13 16 19	23 26 29 22 25 29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3 6 9 3 6 9	12 15 19 12 14 17	22 25 28 20 23 26
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3 6 9 3 6 8	11 14 17 11 14 17	20 23 26 19 22 25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3 6 8 3 5 8	11 14 16 10 13 16	19 22 24 18 21 23
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	3 5 8 3 5 8	10 13 15 10 12 15	18 20 23 17 20 22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2 5 7 2 4 7	9 12 14 9 11 14	17 19 21 16 18 21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2 4 7 2 4 6	9 11 13 8 11 13	16 18 20 15 17 19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2 4 6	8 11 13	15 17 19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2 4 6	8 10 12	14 16 18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2 4 6	8 10 12	14 15 17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2 4 6	7 9 11	13 15 17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2 4 5	7 9 11	12 14 16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2 3 5	7 9 10	12 14 15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2 3 5	7 8 10	11 13 15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2 3 5	6 8 9	11 13 14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2 3 5	6 8 9	11 12 14
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33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1 3 4	5 6 8	9 10 12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1 3 4	5 6 8	9 10 11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1 2 4	5 6 7	9 10 11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1 2 4	5 6 7	8 10 11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1 2 3	5 6 7	8 9 10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1 2 3	5 6 7	8 9 10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1 2 3	4 5 7	8 9 10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1 2 3	4 5 6	8 9 10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1 2 3	4 5 6	7 8 9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1 2 3	4 5 6	7 8 9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1 2 3	4 5 6	7 8 9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1 2 3	4 5 6	7 8 9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1 2 3	4 5 6	7 8 9
46	6628	6637	6646	5656	6665	6675	6684	6693	6702	6712	1 2 3	4 5 6	7 7 8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1 2 3	4 5 5	6 7 8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1 2 3	4 4 5	6 7 8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1 2 3	4 4 5	6 7 8

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52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1 2 2	3 4 5	6 7 7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1 2 2	3 4 5	6 6 7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1 2 2	3 4 5	6 6 7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1 2 2	3 4 5	5 6 7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1 2 2	3 4 5	5 6 7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1 2 2	3 4 5	5 6 7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1 1 2	3 4 4	5 7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1 1 2	3 4 4	5 6 7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1 1 2	3 4 4	5 6 6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1 1 2	3 4 4	5 6 6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1 1 2	3 3 4	5 6 6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1 1 2	3 3 4	5 5 6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1 1 2	3 3 4	5 5 6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1 1 2	3 3 4	5 5 6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1 1 2	3 3 4	5 5 6
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68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1 1 2	3 3 4	4 5 6
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76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1 1 2	2 3 3	5 5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1 1 2	2 3 3	4 4 5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1 1 2	2 3 3	4 4 5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1 1 2	2 3 3	4 4 5
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81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1 1 2	2 3 3	4 4 5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1 1 2	2 3 3	4 4 5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1 1 2	2 3 3	4 4 5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1 1 2	2 3 3	4 4 5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1 1 2	2 3 3	4 4 5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1 1 2	2 3 3	4 4 5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0 1 1	2 2 3	3 4 4
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92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0 1 1	2 2 3	3 4 4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0 1 1	2 2 3	3 4 4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0 1 1	2 2 3	3 4 4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0 1 1	2 2 3	3 4 4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0 1 1	2 2 3	3 4 4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0 1 1	2 2 3	3 4 4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0 1 1	2 2 3	3 4 4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0 1 1	2 2 3	3 3 4

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.20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0 1 1	1 2 2	3 3 3
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.25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0 1 1	2 2 2	3 3 4
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.30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0 1 1	2 2 3	3 4 4
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.33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0 1 1	2 2 3	3 4 4
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35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1 1 2	2 3 3	4 4 5
36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1 1 2	2 3 3	4 4 5
37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1 1 2	2 3 3	4 4 5
.38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1 1 2	2 3 3	4 4 5
39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1 1 2	2 3 3	4 5 5
.40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1 1 2	2 3 4	4 5 5
41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1 1 2	2 3 4	4 5 5
42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1 1 2	2 3 4	4 5 6
43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1 1 2	3 3 4	4 5 6
44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1 1 2	3 3 4	4 5 6
.45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1 1 2	3 3 4	5 5 6
46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1 1 2	3 3 4	5 5 6
47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1 1 2	3 3 4	5 5 6
48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1 1 2	3 4 4	5 6 6
.49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1 1 2	3 4 4	5 6 6

	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
.50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1 1 2	3 4 4	5 6 7
51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1 2 2	3 4 5	5 6 7
52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1 2 2	3 4 5	5 6 7
53	[illegible]	3396	3404	3412	3420	3428	3436	3443	3451	3459	1 2 2	3 4 5	6 6 7
54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1 2 2	3 4 5	6 6 7
.55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1 2 2	3 4 5	6 7 7
.56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1 2 3	3 4 5	6 7 8
57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1 2 3	3 4 5	6 7 8
58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1 2 3	4 4 5	6 7 8
59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1 2 3	4 5 5	6 7 8
.60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1 2 3	4 5 6	6 7 8
61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1 2 3	4 5 6	7 8 9
62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1 2 3	4 5 6	7 8 9
63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1 2 3	4 5 6	7 8 9
64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1 2 3	4 5 6	7 8 9
.65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1 2 3	4 5 6	7 8 9
66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1 2 3	4 5 6	7 9 10
67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1 2 3	4 5 7	8 9 10
68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1 2 3	4 6 7	8 9 10
69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1 2 3	5 6 7	8 9 10
.70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1 2 4	5 6 7	8 9 11
71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1 2 4	5 6 7	8 10 11
72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1 2 4	5 6 7	9 10 11
73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1 3 4	5 6 8	9 10 11
74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1 3 4	5 6 8	9 10 12
75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1 3 4	5 7 8	9 10 12
76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1 3 4	5 7 8	9 11 12
77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1 3 4	5 7 8	10 11 12
78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1 3 4	6 7 8	10 11 13
79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1 3 4	6 7 9	10 11 13
.80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1 3 4	6 7 9	10 12 13
.81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2 3 5	6 8 9	11 12 14
82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2 3 5	6 8 9	11 12 14
83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2 3 5	6 8 9	11 13 14
84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2 3 5	6 8 10	11 13 15
85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2 3 5	7 8 10	12 13 15
86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2 3 5	7 8 10	12 13 15
87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2 3 5	7 9 10	12 14 16
88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2 4 5	7 9 11	12 14 16
.89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2 4 5	7 9 11	13 14 16
90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2 4 6	7 9 11	13 15 17
91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2 4 6	8 9 11	13 15 17
92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2 4 6	8 10 12	14 15 17
93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2 4 6	8 10 12	14 16 18
94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2 4 6	8 10 12	14 16 18
.95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2 4 6	8 10 12	15 17 19
96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2 4 6	8 11 13	15 17 19
97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2 4 7	9 11 13	15 17 20
.98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2 4 7	9 11 13	16 18 20
99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2 5 7	9 11 14	16 18 20

Answers

Exercises 1.1

1. (i) $\in$ (ii) $\notin$ (iii) $\notin$ (iv) $\in$ (v) $\in$ (vi) $\notin$
2. (i) $A = \{-3, -2, -1, 0, 1, 2, 3, 4, 5, 6\}$
   (ii) $B = \{1, 2, 3, 4, 5\}$
   (iii) $C = \{17, 26, 35, 44, 53, 62, 71\}$
   (iv) $D = \{1, 3, 5\}$
   (v) $E = \{M, A, T, H, E, I, C, S\}$
   (vi) $F = \{S, E, T\}$
3. (i) $\{x : x \text{ is a multiple of } 3 \text{ and } x \leq 12\}$
   (ii) $\{x \cdot x = 2^n, n \in N \text{ and } n \leq 5\}$
   (iii) $\{x . x = 5^n, n \in N \text{ and } n \leq 4\}$
   (iv) $\{x \;\; x \text{ is a vowel in the English alphabet}\}$
   (v) $\{x \;\; x \text{ is an odd natural number}\}$
   (vi) $\{x : x \text{ is an even natural number}\}$
   (vii) $\{x : x \text{ is a square of natural number}, x \leq 10\}$
4. (i) $\Leftrightarrow$ (c), (ii) $\Leftrightarrow$ (a), (iii) $\Leftrightarrow$ (d), (iv) $\Leftrightarrow$ (b)
5. (i) Yes (ii) No (iii) Yes (iv) No

Exercises 1.2

1. Yes, No.
2. (i) $\subset$ (ii) $\not\subset$ (iii) $\subset$ (iv) $\not\subset$ (v) $\not\subset$ (vi) $\subset$ (vii) $\subset$
3. (i) False (ii) True (iii) True (iv) True (v) False (vi) True
4. (i) False (ii) False (iii) True (iv) False (v) False (vi) True
   (vii) False (viii) False (ix) True (x) False (xi) False (xii) False
   (xiii) False (xiv) False (xv) True (xvi) True (xvii) False (xviii) True
5. (i) $\varnothing, \{a\}$ (ii) $\varnothing, \{a\}, \{b\}, \{a, b\}$
   (iii) $\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}$
   (iv) $\varnothing$

Exercises 1.3

1. (i) $A \cup B = \{a, e, i, o, u, b\}$
   (ii) $X \cup Y = \{1, 2, 3, 5\}$
   (iii) $A \cup B = \{x : x = 1, 2, 4, 5 \text{ or a multiple of } 3\}$
   (iv) $A \cup B = \{x : 1 < x < 10, x \in N\}$
   (v) $A \cup B = \{1, 2, 3\}$
2. Yes, $A \cup B = \{a, b, c\}$ 3. B 4. A
5. (i) $A \cap B = \{a\}$ (ii) $X \cap Y = \{1, 3\}$
   (iii) $A \cap B = \{3\}$
6 (iii) 7. Yes 8. $\varnothing$

Exercises 1.4

1. (i) $A' = \{1, 3, 5, 7, 9\}$ (ii) $B' = \{2, 4, 6, 8\}$
   (iii) $C' = \{1, 4, 6, 8, 9\}$ (iv) U (v) $\varnothing$
2. A is the set of all natural numbers which are not composite.
3. $(A \cup C)' = A' \cap C' = \{1, 9\}$
   $(A \cap C)' = A' \cup C' = \{1, 3, 4, 5, 6, 7, 8, 9\}$
4. (i) True (ii) True (iii) False (iv) False (v) True (vi) True
5. A' is the set of all equilateral triangles.

Exercises 1.6

1. 2 2. 8 3. 22 4. 30 6. 19 7. 25, 35 8. 60

Exercises 2.1

2. $\frac{3}{8}, \frac{5}{12}, \frac{11}{24}$ 3. $\frac{13}{48}, \frac{7}{24}, \frac{5}{16}, \frac{29}{96}$
4. 0.01, 0.02, 0.03, 0.04, 0.05, . . .
5. $\frac{-23}{100}, \frac{-24}{100}, \frac{-25}{100}$
6. (i), (ii), (iv) and (v)
7. $.\overline{142857}, .\overline{285714}, .\overline{428571}, .\overline{571428}$,
   $.\overline{714285}, .\overline{857142}$

Exercises 2.2

2. (i) 2 (ii) 6 (iii) 14
3. (i) = (ii) > (iii) > (iv) >
4. $|a|^2 + |b|^2 - 2|a||b|$, when $|a| = |b|$
6. 2 24, 2 64 9. .221, . 222
10. (i) Irrational (ii) Rational (iii) Irrational (iv) Irrational
11. 2.101001000100001 . . ., 2 201001000100001 . .
12. .1010010001 . ., 1101001000100001 . .
13 $\sqrt{2\ 1}, \sqrt{2.2}$

Exercises 2.3

1. (i) R (ii) $\{x . x \geq 3, x \in R\}$
(iii) R (iv) $\{x . 3 \leq x \leq 10, x \in R\}$
(v) $\{x : 0 \leq x \leq 8, x \in R\}$
2. (i) $f(1) = 7$, $f(2) = 10$, $f(-1) = 1$
(ii) Not defined (iii) $f(1) = \sqrt{3}$, $f(2) = \sqrt{6}$
$f(-1) = \sqrt{3}$ (iv) Not defined (v) $f(1) = 1 + \sqrt{7}$,
$f(2) = \sqrt{2} + \sqrt{6}$
4. (i) False (ii) True (iii) False
(iv) False (v) False (vi) False
5 . (i) First (ii) Third (iii) Fourth
(iv) Third (v) Second

Exercises 3.1

1. $\sqrt{150}$ 2. $\sqrt[5]{32}$ 3. $\sqrt[4]{405}$
4. $\sqrt{300}$ 5. $\sqrt{\frac{128}{9}}$ 6. $\sqrt{\frac{9}{2}}$ 7. $4\sqrt{5}$
8. $2\sqrt[3]{9}$ 9. $2\sqrt[5]{9}$ 10 $15\sqrt{6}$ 11. $2\sqrt[5]{10}$ 12. $15\sqrt[3]{5}$

Exercises 3.2

1 (i) $\sqrt[5]{3}$ (ii) $\sqrt[4]{10}$ (iii) $\sqrt[3]{4}$ (iv) $\sqrt[3]{6}$
(v) $\sqrt[4]{6}$ (vi) $\sqrt[3]{3}$
2 . (i) $\sqrt{3}, \sqrt[3]{4}, \sqrt[4]{5}$ (ii) $\sqrt[5]{4}, \sqrt[4]{3}, \sqrt[3]{2}$
(iii) $\sqrt[3]{6}, \sqrt[4]{10}, \sqrt{3}$ (iv) $\sqrt[3]{2}, \sqrt[6]{3}, \sqrt[9]{4}$

Exercises 3.3

1. $25\sqrt{2}$ **2.** $5\sqrt{3}$ **3.** $8\sqrt{3}$ **4.** $5\sqrt{2}$
5. $\sqrt{5}$ **6.** $-5\sqrt{2}-20\sqrt{3}$ **7.** $11\sqrt[3]{4}$
8. $3\sqrt[3]{5}$ **9.** $\frac{203}{9}\sqrt{3}$ **10.** 0

Exercises 3.4

1. $7\sqrt{6}$ **2.** $\sqrt{105}$ **3.** $2\sqrt[3]{11}$ **4.** $168\sqrt{2}$
5. $\sqrt[6]{500}$ **6.** $\sqrt[12]{432}$ **7.** $\frac{8}{3}$ **8.** $\sqrt[3]{\frac{1}{3}}$ **9.** $2\sqrt[3]{3}$ **10.** 2

Exercises 3.5

1. (i) $\sqrt{2}$ (ii) $\sqrt{10}$ (iii) $\sqrt{3}$ (iv) $\sqrt[3]{25}$
(v) $\sqrt[3]{6}$ (vi) $\sqrt[3]{2}$

2. (i) $\frac{2\sqrt{5}}{5}$ (ii) $\frac{2\sqrt{3}}{9}$ (iii) $\frac{\sqrt{3}}{6}$ (iv) $\frac{\sqrt{10}}{5}$
(v) $\frac{2\sqrt{77}}{11}$ (vi) $\sqrt[3]{15}$

3. (i) .707 (ii) 577 (iii) .316 (iv) 1.080
(v) .155 (vi) .655

Exercises 3.6

1. (i) $a = 2, b = -1$ (ii) $a = \frac{11}{7}, b = \frac{6}{7}$
(iii) $a = 11, b = -6$ (iv) $a = \frac{9}{11}, b = \frac{19}{11}$
(v) $a = 4, b = 1$ (vi) $a = 2, b = -\frac{5}{6}$

2. (i) $\frac{31+10\sqrt{6}}{19}$ (ii) $6-\sqrt{35}$ (iii) $\frac{47+21\sqrt{5}}{2}$
(iv) $\frac{18-3\sqrt{15}+2\sqrt{10}-4\sqrt{6}}{19}$ (v) $\frac{9+4\sqrt{30}}{21}$
(vi) $\frac{114-41\sqrt{6}}{30}$

3. (i) $\dfrac{1-\sqrt{2}+\sqrt{5}+\sqrt{10}}{12}$ (ii) $-\dfrac{1}{4}(2+\sqrt{2}+\sqrt{6})$

(iii) $\dfrac{1}{60}(6\sqrt{5}+5\sqrt{6}+\sqrt{330})$ (iv) $\dfrac{1}{84}(7\sqrt{6}+6\sqrt{7}+\sqrt{546})$

4. (i) 0 (ii) 1 5. (i) $-.213$ (ii) 14.268

Exercises 3.7

1. (a) $\sqrt{3}$ (b) $\sqrt{3}$, $\sqrt[4]{10}$, $\sqrt[3]{6}$
2. (a) 0 (b) 20 3. 1.732
4. (i) $\sqrt[3]{4}$ (ii) $\dfrac{1}{10}\sqrt[3]{75}$
5. $\dfrac{1}{60}(6\sqrt{5}+5\sqrt{6}-\sqrt{330})$ 6. 0
7. -1.466 8. 8 9. $a=-\dfrac{61}{29}, b=-\dfrac{24}{29}$
10. 0.102

Exercises 4.1

1. (i), (ii) and (vi)
2. (i) $x^7-2x^6-3x^5+\frac{4}{3}x^2+\sqrt{2}x+4$
   (ii) $x^5+2x^4+x^2-\sqrt{3}x$
   (iii) $u^3+u^2-u-\sqrt{2}$
   (iv) $-2y^4-y^3+y^2+3y+4$
3. Monomials : (i), (vi) and (vii)
   Binomials : (ii) and (v)
   Trinomials : (iii) and (iv)

Exercises 4.2

1. (i) 1 (ii) 0 (iii) 0 (iv) 0
   (v) 2 (vi) 4 (vii) 4 (viii) 2
2 (a) $2x^3-8x^2+3x+3$, 3 (b) 4, 0
   (c) $y^6-2y^4+y^3+2y^2-6$, 6
   (d) t^3+2t^2+4t-3, 3 (e) $u+13$, 1
3. (a) x^3-4x^2+x+2, 3
   (b) $u^7-4u^6+4u^2+u+6$, 7
   (c) $-3y^2-y+1$, 2 (d) $8t+4$, 1

4. $2u^4 - 4u^3 - 3u^2 + 11u + 4$
5. $-x^4 + 2x^2 + 2$ 6. $x^3 - 2x^2 + 4x$
7. (i) $x^2 + x - 6, 2$ (ii) $x^3 - 6x^2 + 12x - 8, 3$
(iii) $x^4 + 6x^3 + 12x^2 + 19x + 2, 4$
8. $u^5 + 2u^4 - 2u^3 - 3u^2 + 12u + 4$

Exercises 4.3

1 $\frac{5}{2}x + \frac{3}{2}$, 1, No
2. $y^2 - 2y + 2$, 4, No
3. $u^2 + 5u - 2$, 0, Yes
4. $x^2 - 12x + 13$, -34, No
5 $y^3 - 1$, $3y^2 + 11$, No
6. $t - 1$, 0, Yes
7. $x^3 - 4x^2 + 19x - 65$, $227x + 133$, No
8. $u^2 + u + 1$, $u + 1$, No
9. $2x + 2a - 3$, $2a^2 - 3a + 5$, No

Exercises 4.4

1. (i) -1 (ii) 20 (iii) -13 (iv) 212 (v) -52 (vi) $-\frac{25}{4}$
2. 1 3. (i) 0 (ii) 0 (iii) 60
4. (i) $\frac{2}{5}$ (ii) $\frac{7}{6}$ (iii) $\frac{3}{2}$
5. (i) $-\frac{4}{3}$ (ii) 0 6. (i) -1 (ii) $-\frac{1}{3}$

Exercises 5.1

1. (i) $(2x + 3)(2x + 3)$ (ii) $(3x - 4)(3x - 4)$
(iii) $(12y + 1)(12y + 1)$ (iv) $(6x + 5)(6x + 5)$
(v) $(10 + 3z)(10 - 3z)$ (vi) $(7 + 8k)(7 - 8k)$
(vii) $(a + 5b)(a - 5b)$ (viii) $(2a - 2b + c)(2a + 2b - c)$
(ix) $2(3ax + 4)(3ax - 4)$ (x) $3xy(x + 9y)(x - 9y)$
2. (i) $(x + y - 3)(x - y + 3)$ (ii) $(5x + 6z - 1)(5x - 6z - 1)$
(iii) $x(x + 1)(x - 1)$ (iv) $(1 + a + b)(1 - a - b)$
(v) $(x + 5y)(x + 5y)$ (vi) $(a + b)(a + b + 2c)$

Exercises 5.2

1. (i) $(x + 9)(x + 5)$ (ii) $(x - 12)(x - 10)$
(iii) $(x - 14)(x + 3)$ (iv) $(x - 12)(x - 9)$
(v) $(u - 18)(u - 12)$ (vi) $(y + 3)(y - 1)$
(vii) $(x + 6)(x + 8)$ (viii) $(y - 7)(y + 3)$
(ix) $(x + 9)(x - 4)$ (x) $(x - 11)(x - 12)$

Exercises 5.3

1 (i) $(x - 2)(3x + 4)$ (ii) $(3x + 2)(2x - 3)$
(iii) $(u - 2)(3u - 4)$ (iv) $(2u + 3)(3u + 4)$
(v) $(2y - 7)(y + 3)$ (vi) $(4x - 3)(3x - 4)$
(vii) $(2x + 3)(7x - 1)$ (viii) $(5x - 3)(6x + 5)$
(ix) $(8x - 3)(3x - 7)$ (x) $(x + 1)(5x - 21)$
(xi) $(y + 3\sqrt{3})(\sqrt{3}y + 2)$ (xii) $\frac{1}{2}(x - 4)(x - 2)$
(xiii) $(4x - \sqrt{3})(\sqrt{3}x + 2)$ (xiv) $(7x + 1)(2x + 1)$
(xv) $(px - 3q)(x + 4p)$

Exercises 6.1

1. (i), (ii) , (iv) , (vi) and (viii)

Exercises 6.2

1. (i) Yes (ii) No (iii) Yes (iv) No
(v) Yes (vi) No (vii) No

2. (i) 4 (ii) 6 (iii) 3 (iv) $2(1 + \sqrt{3})$
(v) $\frac{7}{20}$ (vi) $\frac{2\sqrt{5} + 4}{\sqrt{5} + 5}$ (vii) $\frac{9}{4}$ (viii) $2(1 + \sqrt{2})$

Exercises 6.3

1. 21 2. 30 3. 42, 6
4. 12, 36 5. Rs 90000, Rs 45000, Rs 30000
6. 12.5 cm, 7.5 cm 7. 5 cm, 1 cm
8. 90°, 40°, 50° 9. 30, 20

10. 63 **11.** 40 years, 10 years **12.** 30m, 10m
13. 120, 100, 135 **14.** 60 **15.** 3 km/hour
16. 731, 89 **17.** 168, 132 **18.** 54, 81, 108
19. Rs 60000 **20.** Rs 240 **21.** 24 days
22. 80 km **23.** 3.75 km **24.** 1075
25. 400g, 300g **26.** Rs 22000, Rs 13000 **27.** 48
28. 18 km **29.** 320 km/hour, 360 km/hour **30.** 128 ml

Exercises 7.1

1. **(i)** $\frac{2^{\frac{1}{4}}}{2}$ **(ii)** $\frac{10^{\frac{1}{2}}}{100}$ **(iii)** $4\,4^{\frac{2}{3}}$ **(iv)** $\frac{2}{5}(10)^{\frac{1}{3}}$

(v) $\frac{\sqrt{2}}{2}$ **(vi)** $\frac{2^{\frac{1}{2}}}{2^{10}}$

2. **(i)** $\frac{y^{\frac{3}{2}}}{x}$ **(ii)** $\frac{1}{x^{\frac{4}{3}}y}$ **(iii)** $\frac{1}{x^{\frac{15}{2}}}$ **(iv)** $\frac{y^{\frac{9}{4}}}{x^{\frac{5}{6}}}$

(v) $\frac{y^{\frac{1}{6}}}{x^{\frac{2}{3}}}$ **(vi)** $x^{\frac{1}{6}}$

Exercises 7.2

1. **(i)** $\log_2 32 = 5$ **(ii)** $\log_{10} 1000 = 3$

(iii) $\log_3 81 = 4$ **(iv)** $\log_5 625 = 4$

(v) $\log_{10} .1 = -1$ **(vi)** $\log_7 49 = 2$

2. **(i)** $5^2 = 25$ **(ii)** $3^5 = 243$ **(iii)** $10^3 = 1000$

(iv) $2^6 = 64$ **(v)** $4^3 = 64$ **(vi)** $10^{-2} = .01$

Exercises 7.4

1. **(i)** 3.123×10^0 **(ii)** 3.123×10^1 **(iii)** 3.123×10^2

(iv) 3.123×10^3 **(v)** 3.123×10^5 **(vi)** 3.123×10^{-1}

(vii) 3.123×10^{-2}

2. **(i)** 5600 **(ii)** .1436 **(iii)** .024

(iv) 963200 **(v)** 120.56 **(vi)** 012056

Exercises 7.5

1. (i) 3 1038 (ii) 1.1038 (iii) 2.6350 (iv) 1.0504
   (v) $\overline{1}.0993$ (vi) $\overline{3}$ 0792 (vii) $\overline{5}$ 1396

Exercises 7.6

1. (i) 6812 (ii) .8451 (iii) .5011 (iv) 5014
   (v) $\overline{1}.3711$ (vi) $\overline{1}.3718$
2. (i) $\overline{2}.8854$ (ii) $\overline{3}$ 3979 (iii) $\overline{3}.9395$ (iv) $\overline{3}.9795$
   (v) $\overline{3}.7482$ (vi) $\overline{2}.4579$
3. (i) 5 649 (ii) 2 000 (iii) 3 489 (iv) 564.9
   (v) .2000 (vi) .03489
4. (i) 1 + .2086, 16.16 (ii) −2 + .7916, .06189
   (iii) −3 + .5675, .003694 (iv) −4 + .3568, .0002274
   (v) 2 + .5674, 369 3 (vi) −1 + .38, 2399

Exercises 7.7

1. (i) 6331 (ii) .009645
2. (i) 584 9 (ii) .0002497
3. (i) .3794 (ii) 6.508
4. 3.63 5. 19.264, 20 digits
6. (i) 13.92 (ii) 575.8 (iii) 32.18 (iv) 14.15
7. (a) .4391 (b) .0007787 (c) 9.077 (d) 2.118
8. (a) 116.2 (b) .0003889 (c) 16.34 (d) .1744
   (e) 4154 (f) 675.7 (g) .02512 (h) 366.0
9. (a) .3467 (b) 8.106
10. (a) 1 865 (b) 66.36 (c) 15.95 (d) 1.047

Exercises 7.8

1. Rs 148 40 2. 14.2 years 3. 10.4%
4. (a) Rs 81210 (b) Rs 59190 (c) Rs 29050 (d) 7.6 years
5. 40.9% 6. 322 years 7. 645.5 m
8. 1.677g 9. 454.5 cm^2 10. 11.99 m